(4) "Internal reasons for unification" by Professor C. E. COMSTOCK, Brooklyn Polytechnic Institute.

(5) General discussion, led by Professors F. B. WILLIAMS, Clark University, K. D. SWARTZEL, University of Pittsburgh, and C. H. YEATON, Oberlin College.

1. The gist of Professor Young's paper will appear in an early issue of the MONTHLY.

2. The purpose of Professor Karpinski's paper was to show that the greatest teachers of the world's history employed methods, materials, and illustrations to enrich their instruction, as far as such material was available. Even such a text as Euclid's *Elements* contains, in addition to the seventh to tenth books which are strictly theoretical arithmetic, a wealth of algebraic material. Similarly Archimedes in his *Sphere and Cylinder* solves by the intersection of conics a cubic equation and gives what corresponds to analytical conditions for the existence of real roots of a given cubic. Ptolemy, the great geographer-astronomer, introduces into his astronomical treatise whole sections on geometry and on the computation of chords. The trigonometry in Ptolemy's *Almagest* is developed as completely as it would be in a treatise devoted strictly to mathematics rather than to astronomy. With the great Arabic text-book writers and scientists, with Al Khowarizmi and Omar al Khayyam, in particular, the treatises on algebra were enlivened and enriched by a mass of geometrical and practical illustrations. The same was true with Regiomontanus and with such writers as Euler and Newton. Our texts, prepared for a more immature audience, have made the great error of insistence upon strictly logical unity, rather than upon psychological adaptation to the readers for whom the texts are prepared. Historically, at least, good authority can be found for a broader treatment on a unified basis.

3. Taking as the key principle of unified mathematics for college freshmen the statement that subject matter and methods of instruction should be determined by the present and future needs of students who take no more mathematics, Professor Burgess urged that a searching examination of the various activities of life must be made to bring out just what mathematical attainments are really needed. Casual and uncritical statements, such as that of the graduate who says he never uses mathematics, are not to be accepted as evidence without scrutiny. Professor Burgess then suggested a division of the uses of mathematics into three main classes,—purely mathematical or geometrical, experimental, and statistical; and urged that increased attention be paid to the last type as the one which is most frequently needed subsequently by students. He urged that mathematicians recognize statistical mathematics as the mathematics of aggregates in which the individual elements differ from each other to a significant extent, and that they look upon even elementary statistical methods as worthy of serious attention. The statistical topics suggested for the freshman course are percentage analysis, frequency distributions, simple averages and diagrams, index numbers, and a rough determination of trend lines. Except

for the introduction of the statistical point of view, he joined with others in regarding the function concept as a unifying principle. In regard to the methods to be used, he suggested that extreme emphasis on securing expert facility is unwise, that proofs are of secondary importance in this course, that the needs of the individual class should modify the conduct of the course, and that problems should be selected from a wide field.

4. Professor Comstock discussed the four subjects, college algebra, trigonometry, analytic geometry and calculus. The central ideas of these subjects may be stated as follows: college algebra, a complex of somewhat unrelated topics, such as the solution of equations and the transformation of expressions containing the common functions of algebra; trigonometry, the algebra of the circular functions and the arithmetic of the triangle; analytic geometry, characterized by its three problems, finding the locus of an equation, finding the equation of a locus, and finding the properties of conics by algebraic methods; and calculus, concerned with the changes of functions due to changes in the independent variables. Now there has steadily developed an interpenetration of the fields which surround these central ideas, and this may be considered in two aspects: the more advanced subjects must build upon material from the lower courses; certain topics and methods of one subject are found helpful in presenting the concepts of another subject. For example, calculus calls on geometry for illustration; and geometrical concepts are found exceedingly helpful in presenting analytic processes; also the graph has invaded algebra and trigonometry. In fact, a rudimentary unification of all these subjects is actually going on.

The combination of elementary algebra and plane geometry in one course is not desirable, since they are two cultures made up of two sets of distinctive habits which the student must acquire; the exposure to each must be long enough and sufficiently free from interruption for the ideas to set firm as distinct habits. College students, however, are more mature and have these two lines of thought fixed. In separate college courses much time is lost on dislocated topics and unnecessary methods. Partial fractions should be left until they are needed in integration. A more adequate treatment of series can be given when calculus has been acquired. The introduction of the derivative in analytic geometry improves the treatment of many topics, notably the determination of tangents, the plotting of curves and the study of their properties. The student should be introduced to the most fruitful concepts of mathematics as early as possible. There should be a whole-hearted willingness to use any idea or method that will be most helpful at the time, whether the order is traditional or not.

The function concept runs through all four of the subjects that we have considered and may well be used as a unifying bond, the more so because of its three modes of expression, arithmetical, algebraic and geometric. Sufficient time must be given each topic to afford a deep infusion, and the connection between topics should be made evident. A two-year course is desirable, but students who take one year will have acquaintance with the most valuable methods.

5. Professor Williams said that the whole course should be developed in a

logical way, that it is necessary in all colleges to give the students some review of their previous material. It is not so much a question of time; we need to take them where, on the whole, they are found, and give them as much as can be done during the year. He outlined the course which he has given each year since 1907, a course comprising such topics as: the location of points by coördinates, trigonometric functions and applications, motion of a point in a plane, the intersection of two lines, the different approaches to determinants, the motion of three points on a line, and three lines through a point, with a suggestion of the principle of duality, tangents, limits, introduction to calculus, including its first applications. There should be proper attention paid to symbolic representation so as to avoid confusion in the student's mind on this score. The result of his experience of fifteen years is that this kind of a course is not merely possible but is very valuable.

Professor Swartzel believes that it is possible to write a satisfactory text in uniform mathematics for freshmen, but that this task is still to be done. The teaching of functions in this way does undoubtedly give the students a meaning of the term, but not the general meaning of "function" which we desire, such as is done in the traditional course. There should be a further discussion not merely of the topics which have been dropped in making up this course, but also of the educational side of the question,—a development of the mind toward clear, accurate thinking. The new book for this course should pay attention to the best development of the mind and the incorporating of the new relations which may well be added to the course for freshmen. Professor Swartzel has found it difficult in a course in uniform mathematics to get a sort of examination that shall be satisfactory to the teacher and to the pupils, examinations that will readily and fairly test the student's mastery. Further it is difficult to make the subject-matter coherent, and there is a lack of desirable drill in this course. Do not these difficulties make it evident that we are trying to do too much in a one-year course? He feels that the new style of course is best suited to the mediocre students, who at least appear to do a little better under this plan. A great opportunity lies at hand in the line of segregation of students according to their ability; this will incidentally help in this trial of uniform mathematics.

Professor Yeaton has found that much of the success of the uniform course depends on the enthusiasm of the particular teacher giving the course. He personally feels that this course, including the applications of calculus, is good for students who are sure that they will not continue after the first year, and that it arouses their interest and imagination. The course, however, has the fault of including too many topics, a great difficulty unless we plan a two-year course in its entirety. For the good students it seems quite unfortunate to crowd out the analytic geometry of the first year inclusive of some good serious study of conic sections and the necessary drill in such work.

Mr. Lambert of the Coast Survey raised the question whether the free combination of mathematical methods by the ancients was meant for students of the same grades as those in our colleges. He has found in setting examination ques-

tions for government examinations that questions of a mixed sort have been too difficult and that he had to resort to questions of the traditional sort.

Professor Lennes has seen no way of getting at the questions except by actual tests, and he almost despairs of a result for the reason that one's predisposition toward, say, uniform mathematics, as well as other lines of work, will warp one's judgment even with the best of intentions to be unprejudiced. He has found it hard to get the reaction of the pupil to the three or four pills separately administered in comparison with all given together in one dose.

Professor Plant has felt in opposition to Professor Lennes's point of view that we are getting back to what was the original method of one pill, not four pills in one, and that in his observation the use of various of the present-day texts in uniform mathematics has undoubtedly brought about a satisfactory mental development in the pupils.

Professor Berry has found that a difficulty arises in the case of men training for a special purpose who hold mathematics as a tool, and he concludes that for these the proper course in uniform mathematics has not yet appeared. He has used with success a sort of infiltration process, transferring topics from the traditional order to the point in a course where these are needed.

SECOND SESSION OF THE ASSOCIATION.

(6) "Contradictions in the literature of the group theory," retiring presidential address by Professor G. A. MILLER, University of Illinois.

(7) "An English text of mathematics written about 1810" by Professor ELIZABETH B. COWLEY, Vassar College.

(8) "Impressions of mathematics and mathematical instruction in Italian universities" by Professor VIRGIL SNYDER, Cornell University.

6. The address of the retiring president, Professor Miller, will shortly appear in this MONTHLY.

7. Miss Cowley's paper concerned a three-volume mathematical work in manuscript and will be published in a later issue of this MONTHLY.

8. Professor Snyder described mathematical instruction in Italy. In the preparatory course of twelve years in Italian schools, the range and scope are fairly similar to our own, but the pupils are better trained than ours in mathematics and the languages. In the university, the possible range of studies for a beginner is large, but after the initial selection, the subsequent range is very limited. Those selecting pure mathematics, for example, may have one course in chemistry, or in experimental physics, but with this exception, the entire college course is spent on pure mathematics. During the first year the average student takes fifteen hours, in the second he takes twelve, and in the third and fourth nine each and a thesis. If the term examinations are all passed, and the thesis accepted, and defended in a brief formal examination, the student graduates with the degree of doctor of mathematics. All instruction is by the formal lecture method, frequently not accompanied by any exercise period; there is no

control over attendance; a brief oral examination (twenty minutes) at the end of the year is the only criterion upon which to base the estimate of the quality of the year's work. Projective geometry is begun the first year, and the student starts analytic geometry at the same time; both continue throughout the first year. The calculus is not begun until the second year. Everywhere great emphasis is put on reasoning, and the vehicle most frequently and extensively employed for this purpose is synthetic geometry, in which the students become remarkably skillful. While the Italians have contributed essentially to the development of almost all general branches of mathematics, the present theory of algebraic geometry is preëminently an Italian product, and is perhaps the most important Italian contribution to modern mathematics.

SESSION AT THE RESEARCH LABORATORY OF THE EASTMAN KODAK COMPANY.

(9) "The physical problems involved in photographic research" by Mr. L. A. JONES, of the Research Laboratory.

(10) "Mathematics puzzles as an introduction to investigation" by Professor W. B. CARVER, Cornell University.

(11) "The present status of the formal discipline controversy" by Professor N. J. LENNES, University of Montana.

(12) Inspection of the Research Laboratory.

9. Mr. Jones described the general nature of the problems treated in the Research Laboratory, covering such topics as the following: the amount of energy to be used in producing the photo-chemical effect, the instability of the nitro-cellulose base, the use of dyes in making colored lights, sensitizing agents, etc. As an example of the sort of research carried on in the Laboratory, he showed by graphical representation the nature of the study of the sensitiveness of the ordinary portrait photographic plate, the bichromatic and the panchromatic plates. He made evident, in what at once appeared to those present as a functional dependence, the various factors which determine how perfectly the photographic process reproduces the impression gained by the direct vision of an object.

10. The paper by Professor Carver will appear in a later issue of this MONTHLY.

11. This paper contained a resumé and an evaluation of the arguments and experimental investigations which were supposed to have exploded the doctrine of formal discipline, a doctrine which for practical purposes may be said to deal with the effect of training in one ability upon other abilities. Professor Lennes pointed out that the doctrine of the "faculties of the mind" which was used earlier as the basis for the doctrine of formal discipline is not necessary for the latter. Moreover, the experimental investigations that have been made deal only with peripheral activities in which reflection and judgment have little or no part. Finally it is pointed out that the very simple (necessarily so) experiments that have been made show a considerable carrying over into other activities of the effects of training. Hence it is concluded that for practical purposes the

older doctrine of formal discipline so far from being exploded has not even been disturbed. It does require and is in fact in the process of receiving restatement to make it conform to the newer psychology, but its essential content need not be greatly different from the old doctrine.

12. Following the Thursday morning program and business meeting, members of the Association and other visitors were conducted by various members of the research staff through the Research Laboratory, where they were given some notion of the manifold activities involved in this side of the work of the Eastman Kodak Company's business. Costly apparatus of the most elaborate sort, a large range of chemical, optical and other physical equipment, fills the rooms in what seems to the tyro a crowded manner, but what the Laboratory's results show to be a systematic working plan. Many contacts of interest were made by the visitors on this tour of inspection which was finished all too quickly.

Meeting of the Trustees of the Association.

The following 26 persons and one institution, on applications duly certified, were elected to membership.

To individual membership.

P. L. Armstrong, A.M. (Southwestern Presb. Univ.). Instr., Georgia School of Tech., Atlanta Ga.

Sister M. Bernadita, A. B. (Univ. of Colorado). Head of dept. of math., Loretto Heights Coll., Loretto, Colo.

W. H. Boerckel, A.B. (Pennsylvania). Instr., Georgia School of Tech., Atlanta, Ga.

R. W. Brink, Ph.D. (Harvard). Asso. prof., Univ. of Minnesota, Minneapolis, Minn.

A. H. Cowling, A.M. (Texas). Prof., East Texas State Normal Coll., Commerce, Tex.

Elizabeth S. Dice, A.B. (Texas). Teacher, High School, Dallas, Tex.

H. A. DoBell, A.B. (Syracuse). Instr., Colgate Univ., Hamilton, N. Y.

J. L. Driscoll, A.B. (Washington and Lee). Instr., Georgia School of Tech., Atlanta, Ga.

E. F. Freier, A.B. (Minnesota). Eveleth, Minn.

J. C. Hinton, A.M. (Georgia). Dean, Wesleyan Coll., Macon, Ga.

D. N. Kingery, A.M. Prof., Macalester Coll., St. Paul, Minn.

Sister Laurentine Marie, A.B. (Trinity Coll., Washington). Instr., Emmanuel Coll., Boston, Mass.

O. B. Loewen, A.B. (Bethel Coll.). Head of dept. of math., Ottawa Univ., Ottawa, Kans.

T. R. Long, A.B. (Rochester). Asst., Univ. of Rochester, Rochester, N. Y.

R. L. Newlin, B.S. (Guilford). Instr., Guilford Coll., Guilford College, N. C.

R. L. O'Quinn, A.B. (Louisiana State). Instr., Louisiana State Univ., Baton Rouge, La.

Elsie M. Plapp, M.S. (Chicago). Asst. prof., Hollins Coll., Hollins, Va.

V. E. Pound, Ph.D. (Toronto). Instr., Univ. of Buffalo, Buffalo, N. Y.

R. S. Sheppard, M.A. (Alberta). Prin., High School, Strathcona, Alb., Canada.

G. S. Smith, A.M. (Texas). Head of dept. of math., High School, Port Arthur, Tex.

Virgil Snyder, Ph.D. (Göttingen). Prof., Cornell Univ., Ithaca, N. Y.

G. T. Trawick, B.S. in E.E. (Georgia School of Tech.). Instr., Georgia School of Tech., Atlanta, Ga.

Ann Van Oel, B.S. (Drake Univ.). Teacher, High School, Greene, Ia.

V. H. Wells, Ph.D. (Michigan). Prof., Carleton Coll., Northfield, Minn.

R. I. White, A.B. (Pennsylvania). Instr., Georgia School of Tech., Atlanta, Ga.

E. B. Wilson, B.S. (Chicago). Instr., Georgia School of Tech., Atlanta, Ga.

To institutional membership.

Providence College, Providence, R. I. Pres. William Noon, Official representative.

These names, together with those of all others elected to membership during the past three years, have been printed in the *Register of Officers and Members*, 1922.

The Trustees voted to recommend to the business meeting of the Association the draft of the proposed By-Laws which was presented on Thursday morning.

The resignation of Professor A. A. Bennett as Editor-in-Chief of the MONTHLY was received and accepted, it being understood that his duties terminate at the close of the volume for 1922. Professor W. B. FORD of the University of Michigan was appointed Editor-in-Chief beginning with the issue for January, 1923. Some of the Associate Editors for the year 1923 were appointed, the full list to be completed at the December meeting. Professor C. H. YEATON was appointed Assistant Secretary of the Association. Professor L. C. KARPINSKI was appointed Librarian of the Association, and Professor MARY EMILY SINCLAIR, Assistant Librarian.

The Trustees approved the formation of a Southeastern Section to include Alabama, Florida, Georgia, North Carolina and South Carolina, and possibly Tennessee.

The Trustees also transacted various matters of business concerning the Library of the Association, and the extension of the Association's subvention to the ANNALS.

The President was requested to send the greetings of the Association to Professor Neuberg of the University of Liége, and to the Belgian Mathematical Society whose official organ is *Mathesis*.

BUSINESS MEETING OF THE ASSOCIATION.

A draft of the proposed By-Laws of the Association, which had been considered at great length by the Trustees, was recommended by them to the Association; because of the delay of five months in the printing of the MONTHLY, printed notice of the proposed amendments and of the change in the Articles of Association was sent to the members of the Association one month before the summer meeting, the purpose being to comply with the spirit of the requirement of the By-Laws to give such notice through the official journal. The By-Laws as amended were unanimously adopted at the meeting on Thursday morning, September 7th, 1922, as well as the modifications of the Articles of Association which changed the number of Trustees from nineteen to twenty so as to include a Librarian.[1]

W. D. CAIRNS, *Secretary-Treasurer.*

[1] These By-Laws were printed with the *Register*, 1922, 44-46.

INFINITE AND IMAGINARY ELEMENTS IN ALGEBRA AND GEOMETRY.

By R. M. WINGER, University of Washington.

1. Introduction. Imaginary Points in Analytic Geometry. The traditional treatment of imaginary and infinite elements in algebra and geometry has curiously resisted the reform movement in American textbook writing. While imaginary numbers are receiving perhaps more attention in the algebra books, the imaginary point is still an outlaw.[1] The equation $x^2 + y^2 = 0$ "represents a single point or point circle"[2] while the equation $x^2 + y^2 + 1 = 0$ "represents no locus whatever." This is all the more remarkable in view of the historical fact that it was the geometric representation of imaginaries that gave these numbers their algebraic standing. Probably these same writers would not hesitate to speak of a point in space of four dimensions as a geometric image of a tetrad of numbers although such a point is as impossible to plot as one whose Cartesian coördinates in the plane are imaginary. Now that modern pure geometry has found a way to introduce imaginary points independent of algebra, it would seem absurd not to utilize the algebraic approach which is more elementary.

A reciprocal custom prevails with respect to infinity. Modern geometry has found a line at infinity indispensable as a bond of union between projective and metric properties, to render the principle of duality universally valid and to preserve a (1, 1) correspondence between a figure and its projection. Geometry has also found the number ∞ serviceable in the analytic theory of collineations, the parametric representation of curves and in studying the behavior of curves at infinity. But algebra steadfastly refuses to avail itself of these conveniences. True in the theory of equations in one variable the conditions for infinite roots are frequently given, but no mention is made of infinite roots of simultaneous equations where the idea is most valuable.

Authors of college algebras may hesitate to increase the formidable multiplicity of topics or they may regard the difficulties insuperable to freshmen. But these authors do not scruple to encroach on the fields of analytic geometry and even calculus to an extent that important algebraic material is crowded out altogether.[3] On the other hand, projective geometry has usually been presented synthetically so that any algebraic discussion would appear artificial. Careful writers on both subjects follow the sound practice—under the limitations they

[1] Exception must be made of the excellent book by Snyder and Sisam, *Analytic Geometry of Space*. Even here after discussing imaginary elements the authors lapse into the old form of statement,—that a tangent plane cuts an ellipsoid in a "point ellipse," *e.g.*, rather than in a pair of imaginary lines. Crawley and Evans, *Analytic Geometry*, is the only freshman text that has come to my notice which recognizes imaginary points.

[2] This statement always recalls a bizarre work on *Algeometry* in which the author declares that while $x^2 + y^2 = 0$ is the equation of a *round* point no one has been able to write the equation of a *square* point!

[3] One recent book of 250 pages devotes nearly a third of its space to calculus, both differential and integral.

have prescribed—of telling the truth and nothing but the truth even if it is impossible to tell the whole truth. An occasional author however fails to tell the strict truth, as for example when he says that a linear and a quadratic equation in two variables *always* have two solutions or that two quadratics always have four. And I am not sure that the writer who says that the locus of the equation $x^2 + y^2 = 0$ is a single point does not convey the impression that he is telling a general truth rather than a truth restricted to the real domain. For he invariably uses "point" and "locus" to signify "real point" and "real locus" without troubling himself to explain his meaning.

Certainly in analytic geometry where the streams of algebra and geometry merge there would seem to be the least excuse for these half-truths. For nowhere is the interplay between algebra and geometry more beautifully exemplified than in the theories of infinite and imaginary elements. The college student should come to the study of analytic geometry with a knowledge of imaginary numbers which, thanks to such books as Wilczynski's,[1] have been freed of their stigma and made genuinely real in his experience. Imaginary solutions of equations have been accepted on the same terms as real solutions. He has been taught that a quadratic equation in one variable always has two roots, real or imaginary, distinct or equal. Why then should he be told that certain equations have no loci and that equations with imaginary solutions represent curves which do not intersect, if indeed they represent curves at all? Why not take advantage of the ground already gained in algebra and say that a line and a circle in the plane always meet in two points, real and distinct, real and coincident or conjugate imaginary? The new statement not only serves every purpose of the old "two points, one point, no point" terminology but is actually more descriptive of the relation of the line and circle. A generator of a circular cone is a true example of a line that meets the circle of the base in but one point while the axis of the cone meets the circle in no point.

Absolute coördinates. It will be instructive to consider briefly a system of metrical coördinates in which the conventional rôles of real and imaginary are varied. We refer to the absolute coördinates used with such signal success by Professor Morley and his students in discussing metrical properties. In this system the bilinear axes are the circular rays through the Cartesian origin. If X and Y are the rectangular coördinates of a point, the absolute coördinates x, $\bar{x}$ are defined to be

$$x = X + iY, \qquad \bar{x} = X - iY, \qquad i = \sqrt{-1}. \tag{1}$$

The absolute coördinates, being complex, are represented geometrically in the usual manner on the complex plane. But if plotted separately x and $\bar{x}$ would in general determine two points. To avoid ambiguity we plot only the x, although the $\bar{x}$ cannot be ignored. When the Cartesian coördinates are real the absolute coördinates are in general imaginary. The converse is however not generally true. Neither does the reality of the absolute coördinates imply

[1] *College Algebra with Applications*, Boston, 1916.

reality of the corresponding point. Thus on the line

$$x + \bar{x} = 2, \qquad \text{or} \qquad X = 1, \tag{2}$$

those points are real whose x-coördinates are of the form $1 + iY$. The corresponding $\bar{x}$-coördinates are then $1 - iY$. A single real point of the line has real absolute coördinates, viz., the point (1, 1). But while the number pairs (2, 0), $(3, -1)$, $(1 + \sqrt{2}, 1 - \sqrt{2})$, $(i, 2 - i)$ satisfy the equation of the line, they cannot be represented. For if we plot the x alone the first three points would all lie on the line $x - \bar{x} = 0$ whereas this line cuts (2) only in the point (1, 1). If the algebra and geometry are to correspond, we must say that these pairs of numbers are coördinates of *imaginary* points on the line. In this system we have the following criterion:

A point is real or imaginary according as its absolute coördinates are or are not conjugate complex numbers.[1]

Again the parametric representation of curves in absolute coördinates differs from that in Cartesian coördinates. In the latter system real parameters are assigned to real points. In absolute coördinates, on the other hand, if the coefficients in the equations are real, the parameters of real points will be imaginary in general. Instead of thinking of the parameter as running along a line and ranging through all values, we may suppose the parameter to run around the unit circle in the complex plane, *i.e.*, the parameter t is a complex number of absolute value 1. The conjugate of t is then $1/t$. Thus the parametric equations of the unit circle are

$$x = t, \qquad \bar{x} = 1/t, \tag{3}$$

for the x-coördinate of a point is the same as the parameter. The only real parameters attached to real points of the circle are ± 1 which are cut out by the line $x - \bar{x} = 0$.

We need go no farther in this development here. Enough has been said perhaps to indicate the danger of dismissing imaginary coördinates as representing no points. The introduction of imaginary points into analytic geometry obviates a multitude of exceptions which are distasteful to a geometer who believes that his subject is every whit as good as algebra.[2]

[1] Complex refers to either real or imaginary numbers, a real number being self-conjugate.

[2] One has but to glance through the recent book of Osgood and Graustein, *Plane and Solid Analytic Geometry*, to see how many such exceptions must be noted by an exacting writer. A conspicuous example is the important theorem (pp. 167–8) relating to a pencil of curves $u + kv = 0$. They divide the theorem into two parts according as $u = 0$ and $v = 0$ do or do not intersect. The first part has an exception when $u + kv = 0$ represents a single point (or a finite number of points). The second part reads: "Let $u = 0$ and $v = 0$ be the equations of two non-intersecting curves. Then the equation $u + kv = 0$ $(k \neq 0)$ represents, in general, a curve not meeting either of the two curves." The beautiful positive theorem (part one) is thus replaced by a weak and purely negative theorem devoid of its chief interest. Moreover two exceptions are given (in addition to $k \neq 0$), viz., $u + kv = 0$ may represent (*a*) a single point, (*b*) no locus. Even then (*b*) must stand for such diverse cases as constant $= 0$ and $x^2 + y^2 + 2 = 0$. Finally a special case is given when $u = 0$ and $v = 0$ represent parallel lines. How much more satisfactory, having annexed the infinite and imaginary domain, to be able to say: $u + kv = 0$ (*always*) represents a curve through all the common points of $u = 0$ and $v = 0$, and, if $k \neq 0$ or ∞, through no other

2. The Number Infinity in Algebra. On the other hand, the algebraist who believes his subject is as good as geometry will be distressed at the failure of his fraternity to appropriate the advantage won by the geometer in the introduction of infinite elements. The geometer has found it convenient to postulate a point at infinity on every line or a line at infinity in every plane.[1] The most immediate consequence is that parallel lines may then be defined as lines which meet at infinity.[2] How can we make use of this definition and postulate in algebra? Perhaps the best approach is through homogeneous equations when infinite roots are placed on the same basis as zero roots. But we need not resort to homogeneous equations. We might begin by postulating a number ∞—call it an improper number if you like—which shall be the coördinate, in a Cartesian coördinate system along a line, of the "improper" point at infinity on the line.[3] This number is now clothed with properties to conform to those of its geometric counterpart. Thus the x-intercept of the line

$$y = mx - k \tag{4}$$

is

$$x_0 = k/m. \tag{5}$$

Now if $m = 0$, the line is parallel to the x-axis and cuts it therefore at infinity. Accordingly in virtue of (5) we attribute to ∞ the property

$$k/0 = \infty, \qquad k \neq 0, \tag{6}$$

for if $k = 0$ at the same time, the line (1) coincides with the x-axis and intersects it at every point.[4] It is then but a step to the theorem that the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_rx^{n-r} + \cdots + a_{n-1}x + a_n = 0 \tag{7}$$

has r infinite roots if $a_0 = a_1 = \cdots = a_{r-1} = 0$, so that an equation of apparent degree r may be regarded as an equation of degree n with $n - r$ roots infinite.

Although some writers explain that repeated and imaginary roots must be counted if an equation of degree n is to have n roots, few authors include infinite roots in the enumeration on the ground that the degree of the equation has been

point of either curve. The trivial exception here applies only to the parameter whereas the exceptions in the text restrict the base curves themselves. Again one objects to a definition (p. 312) which confines the polar of a point outside a conic to the line segment cut off by the conic—especially when its equation is given, for the equation certainly represents the whole line. These and kindred exceptions due to the refusal of the authors to countenance imaginary and infinite elements and to assign values to indeterminates mar, in my judgement, an otherwise admirable book.

[1] I wish to insist that this is not projective geometry. Any geometry that isolates a line at infinity and invests it with exceptional properties is not projective. The line at infinity is considered in projective geometry because it serves as a bridge between projective and Euclidean metric properties.

[2] This definition suffices for either the plane or space since two intersecting lines necessarily lie in the same plane.

[3] This is precisely what is done in Veblen and Young, *Projective Geometry*, vol. 1, chapter 7—EDITOR.

[4] Other properties may be assigned by considering the intercepts of the parabola $y = ax^2 + bx + c$, but this will suffice for our present purpose.

reduced. The recognition of infinite roots is however no less essential. For example, the student has learned to expect two solutions of a linear and a quadratic equation even though the roots be coincident or imaginary. Then he encounters two equations like

$$x^2 - 9y^2 = 7, \quad (8) \qquad x = 3y + 1, \quad (9)$$

which on combination lead to

$$9y^2 - 9y^2 + 6y = 6, \tag{10}$$

whence $y = 1$, $x = 4$. Again if he attempts to solve (8) with

$$x = 3y, \tag{11}$$

he will get

$$9y^2 - 9y^2 = 0 = 7, \tag{12}$$

a traditional absurdity. Do (8) and (9) then have a single solution while (8) and (11) have none? This will be the view of his textbook. The orthodox algebraist must say—if he does not avoid the cases in question—that pairs of linear and quadratic equations in two variables and with real coefficients fall into five classes as follows: those with two solutions which are (*a*) real and distinct, (*b*) real and equal, (*c*) conjugate imaginary, (*d*) those with one real solution, and (*e*) those with no solution whatever,[1] while his hyper-orthodox geometrical brother would say that a line cuts a conic in two points, one point or no point. In a sense the algebraic statement is above criticism—it is largely a question of the point of view. But the geometrical statement is not adequate since it affords no criterion for distinguishing a tangent from a line parallel to an asymptote, nor an asymptote from other lines which "do not cut the conic." Furthermore, it places tangent and asymptote in entirely different categories whereas an asymptote might at least be defined as the limit of a tangent. The heterodox but progressive analytic geometer would say that *a linear equation and a quadratic always have two solutions and that a line always cuts a conic in two points*—real or imaginary, distinct or coincident, finite or infinite—thus placing algebra and geometry on an equal footing. Only by recognizing infinite as well as imaginary roots and both imaginary and infinite intersections can this theorem be generalized to two equations (or curves) of degrees m and n.

But we need not go to simultaneous equations and the geometry of two dimensions nor to the infinite region of the plane to find a sound reason for introducing infinite elements. The recognition of infinite elements in the geometry on a curve, which is essentially a geometry of one dimension, is extremely useful. For example, the equation of a circle with radius 1 and center at the origin can be written parametrically in rectangular coördinates

$$\begin{aligned} x &= \cos\theta = (1 - t^2)/(1 + t^2), \\ y &= \sin\theta = 2t/(1 + t^2), \end{aligned} \tag{13}$$

where $t = \tan(\theta/2)$. To find the intersections with the x-axis, we set $y = 0$

[1] If imaginary coefficients are taken into account, he would have even more varieties.

whence $t = 0, x = 1$. But obviously the x-axis cuts the circle in a second point $(-1, 0)$. We must say then (a) there is one point on the circle with no parameter, (b) equations (13) represent a circle exclusive of the point $(-1, 0)$ or (c) the x-axis cuts the circle in a second point with parameter $t = \infty$. Is there anyone who will not admit that on considerations of continuity, generality and elegance the last statement is superior? Isn't it more in harmony with the spirit of modern mathematics and is it any less intelligible to the student?

It frequently happens in Cartesian geometry when curves are written parametrically that the parameter ∞ is attached to a point in the finite part of the plane. Another familiar instance is the folium of Descartes,

$$x = \frac{at}{t^3 + 1}, \qquad y = \frac{at^2}{t^3 + 1},$$

the double point of which is located at the origin but has the parameters 0 and ∞. If θ is the parameter on a curve and x and y are rational functions of $\sin\theta$, $\cos\theta$, $\tan\theta$, then the coördinates of the points on the curve can be expressed as rational functions of $\tan\theta/2$ by means of the relations in (13). There will be one point on the curve with the parameter ∞ and this point is as likely to be in the finite as in the infinite region.

3. Infinite Roots and Inconsistent Equations. The solution of the simultaneous equations:

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0, \end{aligned} \tag{14}$$

may be written

$$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \equiv \frac{A}{C}, \qquad y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \equiv \frac{B}{C}. \tag{15}$$

Obviously the solution is unique unless $C = 0$. If we examine the possibilities when $C = 0$, we recognize two cases: (I) A and B not both $= 0$ and (II) $A = B = 0$.

Case I. $C = 0$, $A \neq 0$. The equations are commonly said to have no solution. There are three characteristic hypotheses depending on the way in which C becomes zero.

(a) $a_1 = b_1 = 0$. Then $A = -b_2c_1$, $B = c_1a_2$. The intercepts of the first line are $-c_1/a_1$ and $-c_1/b_1$ which under our hypothesis become infinite and the equation of the first line takes the form

$$0x + 0y + 1 = 0. \tag{16}$$

In other words the line has moved off to infinity and (16) may be regarded as the equation of the line at infinity. Since A is different from zero $b_2, c_1 \neq 0$ and the other line remains finite. The two lines thus meet at infinity,—at the infinitely distant point on the second line.

(*b*) $a_1 = a_2 = 0$. Then $A = b_1c_2 - b_2c_1$, $B = 0$ and the two lines reduce to $b_1y + c_1 = 0$ and $b_2y + c_2 = 0$. Since $A \neq 0$ the two lines exist and remain distinct. They are manifestly parallel to the x-axis (or one is parallel to the x-axis and the other coincident with it) and they meet therefore at the infinitely distant point of that axis.

(*c*) $a_1/a_2 = b_1/b_2 \neq c_1/c_2$. Barring hypotheses (*a*) and (*b*) the equations assume the form

$$a_1x + b_1y + c_1 = 0, \qquad a_1x + b_1y + c_2 = 0,$$

whose solution is $x = \infty$, $y = \infty$. The lines are parallel and meet at infinity.

Under each hypothesis of case I we have found that the two equations may be interpreted as representing distinct lines which meet in a point at infinity. The coördinates of this point will be a solution of the equations and we no longer have pairs of inconsistent linear equations. Or we may retain the term "inconsistent," introducing the definition: The two linear equations (14) are inconsistent when and only when they represent lines which meet at infinity.

Under case II, $A = B = C = 0$ when the lines coincide and the equations have a single infinity of solutions. If however $a_1 = a_2 = b_1 = b_2 = 0$, the lines coincide with the line at infinity and the equations would then be "inconsistent."

Three linear equations. Similarly the solutions of the three equations

$$a_ix + b_iy + c_iz + d_i = 0, \qquad i = 1, 2, 3, \tag{17}$$

may be written $x = A/D$, $y = B/D$, $z = C/D$. If $D = 0$ but not all the numerators are zero, the values of one, two, or all of the unknowns will be infinite. The equations will then be "inconsistent" as before. Geometrically, the three planes meet in a unique point on the plane at infinity.[1]

If $A = B = C = D = 0$, the equations are dependent and have ∞^1 or ∞^2 solutions according as the planes have a common line or coincide. Such equations however may have "no solution" according to the orthodox view, as, *e.g.*, the three equations

$$ax + by + cz = d_i, \qquad i = 1, 2, 3, \qquad d_1 \neq d_2 \neq d_3. \tag{18}$$

These three planes are parallel and hence have their infinitely distant lines in common. We are led to extend our definition of inconsistent equations as follows: Three non-homogeneous linear equations in three variables, whether dependent or independent, are inconsistent when and only when they represent planes which meet only at infinity.

A general definition of inconsistent equations. These results may be generalized for a set of n linear equations in n variables.[2] Indeed the definition is valid when there are fewer equations than variables. Thus the equations of two parallel planes would be inconsistent. Furthermore any set of equations in

[1] There are three possibilities: (1) one plane cuts the other two and is parallel to their line of intersection, (2) two of the planes are parallel and the third a finite plane, (3) two planes meet in a finite line while the third is the plane at infinity.

[2] This demands of course the assumption that the locus of points at infinity in a linear space S_n of n dimensions is an S_{n-1}.

linear form will be inconsistent under similar conditions. For example, the equations

$$\begin{aligned} ax^2 + by^2 + c_1 &= 0, \\ ax^2 + by^2 + c_2 &= 0, \end{aligned} \qquad c_1 \neq c_2, \tag{19}$$

are linear in x^2 and y^2. Subtracting we should get constant $= 0$ so that the equations are inconsistent in the usual sense. But using formula (15) we find $x^2 = \infty$, $y^2 = \infty$ and we may say that the equations have two pairs of equal roots, *i.e.*, the curves have double contact at infinity.

Two equations need not be of the same degree to be inconsistent. An instance of this already noticed is the equation of a hyperbola and that of its asymptote. Another example is furnished by the equations

$$\begin{aligned} y^3 - xy^2 - x^2 + 2xy &= 0, \\ y^2 - y + x + 1 &= 0, \end{aligned} \tag{20}$$

which, on elimination of x, yield

$$0y^4 + 0y^3 + 0y^2 + 0y - 1 = 0. \tag{21}$$

The equations which should have six solutions thus have no finite solutions and the corresponding curves meet wholly at infinity. We may now formulate a general criterion for inconsistent equations:

r non-homogeneous (dependent or independent) equations in n variables, $r \leqq n$, are inconsistent when and only when the loci of the equations intersect, whether in real or imaginary points, wholly at infinity.

The purpose of this discussion has been not merely to criticize the shortcomings of our current elementary textbooks in respect to imaginary and infinite elements but to illustrate how these fruitful ideas can be employed to enrich both algebra and geometry. The suggestions here embodied are consonant with the spirit that pervades modern mathematics and with sound European tradition. It is my conviction that they could be incorporated into our elementary texts without sacrificing either clearness or rigor and at no greater cost of space than is required to detail the exceptions which they eliminate.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

26. Burckhardt on Modern Teaching.

The number of times the teaching of mathematics has been reformed and the general similarity of view of the reformers are always interesting to the student of the history of the subject. Every day, in educational circles, theories are launched forth that have been common property so long that it is not to be wondered that, in certain schools, the history of education is frowned upon,

ostensibly because "we must face the future" but really because the makers of these theories fear the revelations of the past. One of our perennial discoveries is that mathematics needs to have the waste material eliminated, that the pupil should study only those parts that he will use, and that the best place for this is in some form of laboratory.

Among my autographs is a letter on this subject, written somewhat over a century ago, by Johann Karl Burckhardt.[1] Since he became (1799) a naturalized French citizen, his given name more often appears as Jean Charles. He was a member of the Académie des Sciences and of the Institut (Classe des Sciences physiques et mathématiques) and, although primarily an astronomer,[2] is known for his factor tables and for his considerable interest in pure mathematics. M. Jean-Denis Barbié Du Bocage,[3] a well-known geographer and a member of the Institut, had asked Burckhardt for a plan for the mathematical training of his son, and the result was a letter which, in view of its modern ideas, might well form an appendix to any of the recent reports on the improvement of the teaching of the subject. The following is a translation:

Monday, August 30, 1813

My celebrated Colleague,

I have thought a great deal since yesterday about the best thing to do for your son. It seems to me that there are two points which ought to be considered: First, that it is frequently the case that there are taught for long periods many things which are useless from the practical standpoint and for the purpose which I have in mind for your son, the result being that there is not left sufficient time for the necessary features. Second, that the pupil should not be discouraged in his mathematical studies by any difficulties which he may encounter.

Here, then, is the plan which seems to me the most practicable:

(1) The teacher should present only those features of plane geometry which are necessary for the complete understanding of the theorems which form the foundation of surveying. He should therefore omit such non-essentials as the actual computation of the value of the number $3.14159\cdots$, a thing for which your son would have neither the time nor the need.

(2) After this the instructor should teach algebra up to and including equations of the second degree. The first chapters of Euler[4] form a good model for this kind of work and also include the theory of logarithms. I believe that this study is necessary, if only for the purpose of knowing how to read and use those algebraic formulas which are so necessary in practical mathematics.

(3) Plane trigonometry.

(4) Spherical trigonometry, which should be preceded by certain necessary theorems of geometry which were not taken up in the earlier course. The treatment which M. Delambre has given in his *Abrégé*[5] seems to me to merit preference.

(5) I have not mentioned solid geometry, but it is easy to determine a geographic position[6] even if ignorant that the cone is a third of a cylinder, &c, &c. The teacher may therefore give whatever the time permits.

Perhaps the difficulty will lie in finding a teacher who can carry out this idea. The plan requires more work on his part, fewer [private] lessons and [hence] a diminished financial return. Perhaps, also, he may hesitate to make himself known through fear that his colleagues will decry

[1] Born at Leipzig, April 30, 1773; died at Paris, June 21, 1825.

[2] I have, for example, the original manuscripts of his papers "Sur la cométe actuelle" and "Tables de la Lune" (introduction), presented to the Institut in 1811, each with Delambre's signed memorandum.

[3] Born at Paris in 1760; died in 1825. He founded the Société de Géographie in 1821.

[4] His algebra had appeared at Petrograd (Petersburg) in 1770 and had later been translated into French and English.

[5] *Abrégé d'astronomie* (Paris, 1813) which had just appeared.

[6] Referring to M. Barbié Du Bocage's standing as a leading geographer.

him as disloyal to his profession.[1] On the other hand it is quite possible to lead the teacher to see that this work, once done, will serve to assist others, since circumstances will often force parents to demand more expeditious teaching and such as is directed towards practical ends.

With respect to the necessary equipment, it should be ordered early, since the makers are often engaged for a long time ahead.

Finally, on Nov. 8 and Nov. 9 certain stars will be in eclipse about 6 o'clock in the evening. If your son wishes to observe them with me, he may come about 5 o'clock. If the weather should be unsettled, he would not need to come, for other opportunities will offer themselves.

With sentiments of high consideration, I am,

Your servant and colleague
BURCKHARDT.

It would be interesting to know, but exceedingly wearisome to attempt to find, the number of times that these same ideas have been advanced in the hundred and ten years which have elapsed since Burckhardt "thought a great deal since yesterday" on the great problem of mathematical education. I often wonder if some of the reformers give any more time to the subject than Burckhardt did on this occasion.

27. BURCKHARDT AND THE ETERNAL PROBLEM OF PUBLICATION.

The World War has emphasized the difficulty of publishing scientific works of every kind, both in this country and abroad. The gift of the Hegeler Trust to the Mathematical Association of America is affording relief in one important line at present, the National Research Council has secured funds to enable it to make a beginning in another line, and the Carnegie Institution of Washington has lent a hand, but there still remains the great difficulty of finding some way of publishing existing manuscripts of undoubted mathematical value and of financing the preparation of others.

In view of this state of affairs it is interesting to see occasional evidences of the fact that the problem is not merely one of the present day. It was inadequately solved in earlier times by finding a patron among the old nobility and dedicating to him a work of which he could rarely comprehend even the general nature, but after the French Revolution this method passed rapidly into disuse and the favorite method came to be by resort to communications to learned societies.

Among my autographs is a letter written by Burckhardt[2] in 1814 showing the trouble he was having with his factor table, and it reads not unlike dozens of communications of the present day. The address is mostly obliterated, but it was that of a member of the Institut. The translation is as follows:

Friday, Sept. 30, 1814

Monsieur,

I propose to send, next Monday, a letter to M. Pfaff,[3] professor at Halle, for the purpose of placing in his hands a list of books which the Institut wishes to buy at the sale of the library of M. Klügel.[4] If you wish to purchase anything from this collection, I wish you would send me a note on Monday, at the Institut.

---

[1] In the original, as a *gate-metier*, one who debases his profession.

[2] See page 298, footnote 1.

[3] This was Johann Friedrich Pfaff (1765–1825), who had gone to Halle four years before this.

[4] This was Georg Simon Klügel (1739–1812), whose dictionary of mathematics appeared at Leipzig 1803–1808 (3 vols. with later fourth and fifth volumes).

During the period in which we were so anxious to carry on our scientific pursuits, I worked out the factors of the third and fourth millions.[1] Madame Courcier has offered to print them if the Institut will contribute a thousand francs towards the expenses. This is only 500 francs for each million, although the second million alone has required a sacrifice of 1000 francs. I have a little repugnance at proposing this request; nevertheless I shall gain nothing except the uncomfortable labor of correcting the proofs, which probably no one after my death would ever attempt. On the other hand, I have seen in the reports of the past year that 1500 francs was given for the printing of botanical works, and yet that the mathematical section did not make any expenditure of a similar nature.

Madame Courcier has a good opinion of the sale of this work, since the English have bought several copies of the second million; if this hope is not maintained, she will probably increase her demand to such an extent that it would seem perhaps advisable to profit by the offer which she now makes.

I have the honor of being, with respect,
Monsieur,

V. T. H. & T. O. S,[2]
BURCKHARDT

Below there is a memorandum in Delambre's hand:

"Granted 1000 fr. 7 November 1814."

The letter reads so much like several that have come to my attention in recent years that I feel that it may contain some suggestions of value and some words of encouragement to those who have to meet the problem at the present time.

QUESTIONS AND DISCUSSIONS.

Edited by C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

NEW QUESTION.

The following question was asked by a member of the Ohio Section of the Association, at the meeting of April 20, 1922:

47. Can anyone teaching the theory of investment furnish references for bond valuation formulas when the income rate is a varying function of the time?

DISCUSSIONS.

In the discussion concerning definitions, Professor Allen draws attention to the inexcusable carelessness of many writers on algebra in the terminology of complex numbers. It seems a reasonable demand that an author should use the terms "real," "imaginary" and "complex" in such a way that it may be definitely determined whether or not a given number belongs to one of these classes; and it is rather surprising to learn how few writers of texts have taken the trouble to do this consistently.

The vague and unsettled use of the word "imaginary" in former times can scarcely be blamed for the present situation. It is true that negative numbers

[1] That is, of his *Table des diviseurs pour tous le nombres des premier, deuxième et troisième millions, avec les nombres premiers qui s'y trouvent*, which was published in complete form, at Paris, in 1817.

[2] Votre toute honoré et tout obeissant serviteur.

were at one time called imaginary, and that infinity is sometimes apologetically placed in that category. No doubt also, in popular speech, a cash balance becomes imaginary when it attains the value zero. Still, in careful mathematical speech, a certain definite usage has become established, and departures from this are for the most part the result of negligence.

It is, for instance, generally understood that "imaginary" is equivalent to "unreal," and that "complex" includes the real. When anyone admits these things on one page, and on the next writes "complex" for "imaginary," he is simply following the bad tradition that mars so many older text-books, of *forgetting to allow for the extreme cases.*

It would be easy, but unnecessary, to name some books of this type in which we can never be certain that the positive number system is not going to include zero, and where the inequality $x > y$ is or is not consistent with $x = y$ according to the caprice of the author. In geometry too a great deal of confusion commonly prevails about the limiting cases. Of the many writers who entreat the reader to accept as a triangle one that has a zero angle, how many take the trouble to warn him of the necessity of re-stating some elementary theorems? For instance, a triangle may have two angles equal, but no two sides equal.

The reader will of course not take too literally the "definition" of i as one of the roots of $x^2 + 1 = 0$. How do we know that this equation has two roots, and not more (as in quaternion theory)? The answer is that we have already at this stage decided that a root i exists, and that it shall obey certain formal algebraic laws. It is a consequence of these assumptions that $x^2 + 1 = 0$ only when $x = \pm i$. It is therefore a matter of choice whether we take i to be a one- or a two-valued number.[1]

It may be noted that Professor Allen's last suggestion is exactly the opposite of one that is sometimes followed. For example, Harkness and Morley[2] denote by $\sqrt[n]{a}$ *any* nth root of a, and by a^x the exponential function of $x \log a$, where $\log a$ is the *principal* logarithm and hence a^x is one-valued. This plan has one inconvenient consequence: it makes an odd root of a negative number imaginary. It agrees with the general practice of making e^x one-valued.

The discussion by Professors Cajori and Miller arises from Professor Miller's former article on the same subject, and requires no comment.

I. Definitions of Imaginary and Complex Numbers.[3]

By Edward S. Allen, Iowa State College.

In reading certain parts of about 60 texts on algebra—those of Chrystal, Serret, and Weber among them[4]—I have discovered with surprise that the

[1] For the synthetic treatment of complex numbers as pairs of real numbers, see L. E. Dickson, *Elementary Theory of Equations*, p. 21.

[2] *Introduction to Analytic Functions*, London, 1898, p. 24, 168.

[3] Read at the April meeting of the Ohio Section of the Association.

[4] The list includes also books by all members of that subcommittee of the National Committee on Mathematical Requirements which made the admirable report on "Terms and symbols in elementary mathematics." Dickson's *First Course in the Theory of Equations*, published since this discussion was submitted to the editors of the Monthly, conforms to the policy here recommended in its elegant, brief exposition of the complex number system.

treatment of imaginary and complex numbers is in no instance free from logical inconsistency. A discrepancy is usually found before the following four items have been passed:

1. Definition of imaginary numbers,
2. Definition of complex numbers,
3. If a, b, and c are real, and $b^2 - 4ac$ is negative, then the roots of the equation $ax^2 + bx + c = 0$ are imaginary,
4. The sum of two complex numbers is a complex number.

Probably the most common error, one which persisted in books published as late as 1916, is the definition of an imaginary number as an even root of a negative number. A class of numbers which includes $1 + i$ and $\sqrt{3} + i$, but excludes $2 + i$ and $1 + \sqrt{3}\, i$, is of no value in connection with the usual theorems.

Some authors, realizing the strength of this argument, define imaginary (or pure imaginary) numbers as those whose squares are negative. A complex number s then defined as the sum of a real number and an imaginary (or pure imaginary) one. Real numbers are then not complex; yet the authors using these definitions do not balk at the statement that the sum of two complex numbers is always a complex number. $(2 + 3i) + (4 - 3i) = 6$.

Another group of books handles the subject in the following way. A pure imaginary number is defined as the product of i and of a real number (so that 0 is included). Imaginary and complex numbers are then declared to be identical —sums of real and pure imaginary numbers. The theorem about imaginary (or complex) roots of a quadratic equation loses its meaning, real numbers being, by definition, imaginary. This is Weber's procedure, for instance. Serret, in his *Algèbre Supérieure*, argues in a similar way. He uses the word "imaginaire" on page 86, volume 1, to indicate any number in the complex plane, but on page 269, for instance, to indicate one not on the real axis. Likewise Chrystal, on page 222 of his first volume, intends complex numbers to cover the whole plane; on page 134 ("the coefficients in the factors are complex numbers") he wishes them to avoid the axis of reals. And he uses the word "imaginary" with never a definition.

Two text-books I found, which were so careful as to miss rigor but slightly; perhaps their comparative freedom from error should be specifically mentioned. In Wilczynski and Slaught's *College Algebra* I detected only this flaw—that on page 35 complex numbers must avoid both axes, on page 105 they are excluded only from the real axis, while on page 189 they are allowed to occupy any position on the plane. In Eiesland's *Advanced Algebra* there is but this—on pages 64 and 66 it was forgotten that, according to the definitions used, 0 is real, pure imaginary, and complex, but not imaginary.

Now it is time for "constructive suggestions,"—suggestions which will in no case be new. In the first place let i be defined as one of the roots of the equation $x^2 + 1 = 0$. A pure imaginary number is then the product of i and of

any real number (including 0). A complex number is the sum of a real number and of a pure imaginary one. Finally, an imaginary number is a complex number which is not real. Linguistically, we don't like to have imaginaries come later than pure imaginaries. Very well, say "neomonic," if you wish. The important thing is that there must be a name to cover all numbers of the complex plane, a second one for those on the vertical axis, and a third for all numbers not on the horizontal axis.

Now that I am offering suggestions to writers of text-books and dictionaries, I will venture on another remark. $\sqrt{x}$ is always called a single-valued function when x is positive or zero, it is often regarded as such when x is negative, but it is undoubtedly double-valued when x is—if I may use the definition in the last paragraph—imaginary. $\sqrt[3]{x}$ is single-valued if and only if x is real, $\sqrt[4]{x}$ only if x is positive. It is perhaps inevitable, it is surely bewildering, that the same symbol should indicate, now a single-valued, now a multiple-valued function. There is a need for a symbol which shall always indicate that we may take our choice among all the nth roots of x. Should we not agree that $x^{1/n}$ shall be that symbol? Should not future books say that $4^{1/2} = \pm\sqrt{4} = \pm 2$?

II. The Formula $\frac{1}{2}a\,(a+1)$ for the Area of an Equilateral Triangle.

A Reply to Professor Miller by Florian Cajori, University of California.

In this Monthly (*1921*, 257) Professor G. A. Miller writes on Gerbert's explanation of the question why $\frac{1}{2}a(a+1)$ gives too large a value for the area of an equilateral triangle; Professor Miller claims that Cantor's figure is "inaccurate" and then states:

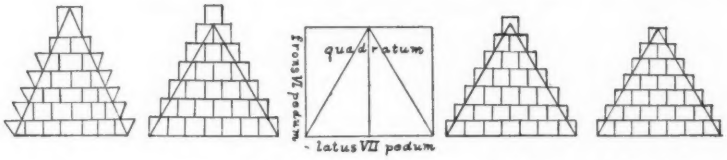
"What is more important is the fact that the corresponding figure found in various histories is still more misleading since it represents according to the explanations in the text an isosceles triangle whose base is equal to the altitude, while the text itself relates to an equilateral triangle. This fact can be verified by consulting either edition of Cajori's *History of Elementary Mathematics*, 1896 or 1917, p. 132."

Professor Miller is in error; the figure in the 1917 edition does not represent "according to the explanations in the text an isosceles triangle whose base is equal to the altitude"; my figure, as well as my explanation, fit (as they should) the case of an equilateral triangle.

One is astonished at Professor Miller's declaration that Bubnov in his edition of Gerbert's *Opera mathematica* gives "a correct figure," and that those of Cantor, Günther and Cajori are all inaccurate or misleading. In the first place, Bubnov gives *two* figures. In the second place, Professor Miller misses completely the essential fact that we do not possess Gerbert's own drawing, and that the drawings in our histories are necessarily conjectural. Neither of Bubnov's two figures agrees with the figure given in Pez's edition of Gerbert's geometry and in the reprint of Pez by Migne in 1880. Pez's figure is due to some unskilful scribe of the Middle Ages whose copy of that geometry contains errors both in the text

and in the figures.[1] In the scribe's drawing (which we here reproduce) the 28 small areas ("XXVIII pedes") are not all alike either in shape or size; hence that drawing is obviously wrong and does not correctly represent Gerbert's reasoning.

Gerbert's text is not sufficiently explicit to enable us to reconstruct the figure with certainty. Three somewhat different attempts have been made, one by Cantor, a second by Günther, a third by Bubnov. We reproduce all three.



From MS. used by Pez | Bubnov's conjectures | Cantor's conjecture | Günther's conjecture

Bubnov's first figure represents Gerbert's rough approximation of 6 feet as the height of the equilateral triangle whose sides are 7 feet. Gerbert knew that 6 was only approximate. Nevertheless Bubnov exhibits the impossible figure in which the sides are 7 and the height is exactly 6.

Cantor's figure shows that the altitude really exceeds 6. For any equilateral triangle whose side is an integer exceeding 7, some of the squares will be wholly outside of the triangle and a very slight but obvious alteration of phraseology in Gerbert's letter would be the only change necessary to make the argument general.

In Günther's drawing, 28 rectangles are used instead of 28 squares. With Gerbert "quadratus" sometimes meant "rectangle."[2] These rectangles are together less in area than $\frac{1}{2}a(a+1)$, nevertheless they more than cover the triangle. The advantage of this construction is that it gives elegance to Gerbert's argument by being at once applicable to any equilateral triangle, be its sides 7 or 30 or a still larger integer. Professor Miller's declaration that Gerbert's argument is "trivial" because it cannot be extended to cases in which $a > 7$ is unwarranted by the facts.

Professor Miller misinterprets Hankel. When Hankel estimates Gerbert's letter as "the first mathematical paper of the Middle Ages which deserves this name," he means the Middle Ages *in Europe*. Hankel had considered Hindu and Arabic research in previous chapters. His statement is explicit; his "Mittelalter" refers, as he says, to the "abendländischen Nationen."

COMMENTS ON THE "REPLY" OF PROFESSOR CAJORI BY G. A. MILLER, University of Illinois.

The main question involved in the preceding "reply" seems to be whether square feet, or feet which are rectangles having unequal sides, were used in

[1] H. Weissenborn, *Gerbert*, Berlin, 1888, p. 42.

[2] See Gerbert's "Geometria" in J. P. Migne, editor, *Patrologiae cursus completus ecclesiae latinae*, vol. 139, Paris, 1880, p. 137. Gerbert, *Opera mathematica*, ed. by N. Bubnov, Berlin, 1899, p. 344.

measuring the areas of the particular equilateral triangles under consideration, since the figures in question relate to the following statement by Gerbert: "Behold in this little diagram there are included 28 feet, not all of which are integral." [1] A little earlier in the same letter Gerbert said: "I believe it is known to you what feet are linear, what square, and what cubic, and that in measuring areas we take only square feet."

One of the two rules given in this letter by Gerbert to find the area of an equilateral triangle was to multiply one half of the numerical measure of the base by the numerical measure of the altitude. There seems to be no question here as regards the kind of feet used, but, as I understand it, the author of the preceding "reply" aims to convey the impression that the figure in the 1917 edition of his *History of Elementary Mathematics* is correct because he used here the term *rectangle*, instead of *square*, to describe one of the 28 feet of which Gerbert spoke. On page 116 of his later second edition of "*A History of Mathematics*" (as well as in a part of the above "reply"), he retained the more definite term "square" for the same purpose, and it seemed only fair to give him credit for concluding finally that the latter term is the better in this connection. This credit seems to have been undue.

If one would like to assume that Gerbert used rectangles of unequal sides as units of measure, one meets with the difficulty of assigning numerical values to these sides as well as to the three concurrent edges of the corresponding rectangular parallelopiped used in measuring solids, to which Gerbert refers in the same letter. It is singular that while the present writer has claimed that Gerbert's letter is of much less scientific value than the references to it in various mathematical histories imply, he does not want to make this letter appear so ridiculous as, it seems to him, the author of the preceding" reply" does, who seems to try to uphold the favorable statements relating to this letter but also to attribute to it by implication a gross want of clearness as regards units of measure.

The foregoing remarks may suffice to make it clear that it is practically certain that Gerbert meant 28 *square feet* when he referred to the diagram to which the figures noted in the first part of the above "reply" relate and hence they seem to justify my claim that most of these figures are inaccurate. It should be noted that by an accurate figure I do not mean that it must be drawn to scale. The sides of the supposed equilateral triangle in the *History of Elementary Mathematics* noted above are not equal, and the squares, or what should be squares, have unequal sides, but I did not find fault with these points. What one seems to have a right to demand of a figure is that if one makes allowances for slight inaccuracies in construction and if one interprets the figure in the light of the subject matter and the text, it should actually illustrate the points in question. In the figure given by Cantor the vertex of the equilateral triangle is about .6 of a unit above the base of the uppermost square while it should be about .06 of a unit above this base. This seemed to me to be too much of a discrepancy to

[1] G. A. Miller, *School Science and Mathematics*, vol. 21, 1921, p. 650.

be regarded as correct while the discrepancy in Bubnov's figure seemed to be too slight to be noticed in a small drawing.

I fail to see why it is assumed in the above "reply" that I missed "completely the essential fact that we do not possess Gerbert's own drawing." It is true that I said that Bubnov gave a correct figure (I did not say that he gave only one figure), but it is to be assumed that the authors of histories in question thought they also gave correct figures, and this assumption is supported by the above "reply." No one, as far as I know, has accused these authors of being ignorant of the fact that their figures are not reproductions of Gerbert's figure. The present writer has never said in public or in private that he thought that we possessed Gerbert's drawing.

The main object that the present writer had in view in making the preceding remarks was to throw some additional light on the nature of the questions involved in the above "reply" to his article. It was there stated that the article "deals admittedly with questions whose satisfactory treatment would require a large amount of space." It can scarcely be supposed or desired that different writers on historical questions should agree as regards all details, since interpretation of language such as the one noted in the last paragraph of the preceding "reply" is often involved, but I hope that enough has now been said to make the main points in question clear to all who may be interested therein.

FINAL NOTE ON THE DISCUSSION BY FLORIAN CAJORI, University of California.

Professor Miller's original contentions are (1) that Bubnov has a "correct" figure, (2) that Cantor's and Günther's are "inaccurate" and "misleading," (3) that Cajori's represents "an isosceles triangle whose base is equal to the altitude," (4) that Gerbert's explanation is "trivial," (5) that Hankel's appreciation of Gerbert's letter is invalid.

Of these only the first two require further comment. My contention that, from Gerbert's text, one cannot tell with certainty what Gerbert's own figure really was, seems borne out not only by the different conjectures made by Cantor, Günther and Bubnov, but also by the fact that Professor Miller, after pronouncing Bubnov's figure "correct," suggested in *School Science and Mathematics*, XXI, p. 652, a different figure of his own. This, however, *does not fit Gerbert's equilateral triangle*. To me, Bubnov's figure is impossible, because it represents to my eye the height as exactly 6. In my judgment, Cantor's figure conveys the truth even to a casual reader by rendering the existence of a fractional excess over 6 strikingly visible. Cantor's and Günther's, though different, appear to me satisfactory illustrations to Gerbert's ingenious letter. In my *History of Mathematics*, 1919, I follow Cantor's explanation; in my *History of Elementary Mathematics*, 1917, I follow Günther's.

Professor Miller says that the main issue is whether the unit of area in Gerbert's letter is the square foot or a rectangle that is not a square. But I have not claimed nor implied that it was other than a square foot. We agree further that "quadratus" sometimes meant "rectangle," for in his own translation of

Gerbert's letter this rendering is given once. His criticism of Günther's figure is valid only if other interpretations of the Latin involved are invalid. Günther's argument, I take it, is simply this, that $\frac{1}{2}a(a+1)$ squares exceed the $\frac{1}{2}a(a+1)$ rectangles drawn, which in turn exceed the area of the given triangle. Finally, I predict that, if Gerbert's own figure is ever brought to light, it will be found to be not Bubnov's nor Miller's, but Cantor's or Günther's.

RECENT PUBLICATIONS.

REVIEWS.

New Mathematical Pastimes. By P. A. MacMahon. Cambridge, University Press, 1921. 8vo. 7 + 116 pages. Price 12 Shillings.

This is an interesting little volume filled with strange and bizarre figures,[1] and punctuated with quaint quotations in verse.[2] The notions are extremely simple (which may surprise readers of some of the author's more serious works) and the language is clear almost to verbiage, as for example (page 50): "The question now before us is the transformation of the set of pieces so that they will no longer be of the same shape and differently coloured or numbered. It proves to be possible to effect this so that the pieces are of different shapes, and are not differently coloured, numbered or otherwise distinguished. In fact, instead of having a set of pieces of the same shape and differently coloured we construct a set of different shapes but of the same colour. The boundary of the assembled pieces now varies in shape with each type and variety but is not otherwise distinguished." The matter is as a whole fresh and pleasingly consecutive. It is far removed in content and treatment from the advanced texts usually issued by the Cambridge University Press.

The author describes his purpose in the following words: "The author of this book has, of recent years, devoted much time and thought to the development of the subject of 'Permutations and Combinations' with which all students are familiar. He has been led, during that time, to construct, for use in the home circle, various sets of pieces, of elementary geometrical shapes based upon these ideas, and he now for the first time brings them together with the object of introducing, in a wider sphere, what he believes to be a pleasant by-path of mathematics which has almost entirely escaped the attention of the well-known writers upon Mathematical Recreations and Amusements. The book differs *in toto* from their works because everything that it contains, with scarcely an exception, is the invention of the author. It is not a bringing together of materials derived from wholly different ideas. From beginning to end it proceeds along one defined

[1] In its few small pages are one hundred thirty-five separate cuts, averaging some ten distinct designs to a cut.

[2] There are fifty-seven poetic quotations from a wide range of classical sources, scattered through the book. Most of these in their original setting could have had but the most tenuous connection with the thought of this text. They serve merely to entertain the reader by their casual verbal allusions.

path from which it never diverges. One continuous thread of thought runs through it from cover to cover."

The book is divided into three parts. The first two are intimately related and yield a systematic method of constructing certain types of what in America we call "jig-saw puzzles." Those here treated have all of their pieces distinct in shape and possess the peculiar advantage of having a minimum number of distinct types of edges. The age-old and ever fresh popularity among children of picture cubes and jig-saw puzzles is ample excuse for the interesting and novel study here presented. A map dissection, in which no two pieces will fit together unless they match in the completed mosaic, will amuse youthful minds but gives training and practice in little more than the simplest sort of visual perception. A puzzle offering the maximum number of possible pairings other than the correct one has an added interest proportional to the ingenuity required. Probably most readers of the MONTHLY have seen the celebrated "Chinese" checkerboard puzzle, for which the checkerboard is sectioned into many differently shaped pieces each of which contains several of the original component squares. This deservedly popular incitement to hours of fruitless patience is not mentioned explicitly by the author. The checkerboard puzzle is exasperatingly baffling, but the reviewer feels convinced that for a given number of pieces the systematic treatment by the author bids fair to yield the most ingenious designs. Perhaps some enterprising toy-maker will forthwith attempt to copyright a large number of these and the present generation may yet greet a whole class of fresh puzzles, new in detail but in general character nearly as old as the human race. The author refrains from touching upon related but diverging problems, such as the dissection of a given figure into an assigned number of pieces which may be refitted to form another given figure.

Part III while following readily upon the previous portions of the book has a wider mathematical appeal, and does not in fact require the elaboration of the first two parts for its understanding. It is concerned with the principles of repeating designs. The treatment is synthetic in that it proceeds from the elementary bases of triangles, parallelograms, squares and hexagons to more elaborate figures. In this third part, one meets many interesting and novel designs but the reviewer is personally disappointed at the lack of suggestive references. The subject is obviously identical in content with the analysis of repeated designs in the Euclidean plane. Far from being new, this is one of the classic lines of investigation in connection with elliptic functions. A possible fundamental region for an elliptic function is no other than a figure whose duplicates under translation may be made to fill the plane without overlapping. The celebrated investigations which in a vastly wider domain have applied group theory to transcendental functions (typified for example by Klein's work on the elliptic modular function) have been left unacknowledged and worse still apparently unutilized by the author. A more definite reduction to the standard types of fundamental triangles might have systematized the discussion. The only excuse although it is perhaps an adequate one for this proofless synthetic treatment is the

fact that the work merely claims to be a pastime. The combinatorial character of the first two parts is suggestive of the analogous problem of the magic square. There one does not expect a completely analytic treatment except under favorable restrictions. The third part deals rather with systematic material affected by the presence of arbitrary arcs. It becomes a problem of analysis situs combined perhaps with certain non-scientific esthetic demands of minor interest. Possibly in the more extensive discussion that the author promises the objections here raised may be no longer applicable.

A useful but limited chronological bibliography of works on mathematical recreations is given in the appendix.

The casual reader may be disappointed in finding no explicit statement of puzzles to tax his ingenuity and arouse his interest. The "pastimes" are suggested rather than directly announced. But any one with a little imagination and with a healthy taste for numerical and tactical puzzles will find a fruitful source of amusement in this book. A far more instructive book might perhaps have been developed upon the same basis of original material but no one should demand instruction in any book claiming to offer nothing more than new mathematical pastimes.

ALBERT A. BENNETT.

NOTES.

The excellent *Revue de l'Enseignement des Sciences*, published by Felix Alcan, Paris, 1909–1920, has ceased publication. Its place is taken by *Bulletin Scientifique des Professeurs de l'Enseignement du 2e degré* (*B.S. 2*) which has been published twice a month since October, 1921, at Chaussenueil, France; P. Martin, director; 12 francs a year.

Mr. ROGER S. HOAR, a member of the Association, has printed privately a discussion of the mechanics of a new design of gasoline power shovel. It is in a simple form, convenient for students of elementary mechanics. Any number of copies gratis are to be had upon application to him, care of Bucyrus Company, South Milwaukee, Wis. This suggestion may appeal to some teacher looking for "practical" applications.

Mr. HARRY B. MARSH, head of the department of mathematics in the Technical High School, and Springfield Junior College, Springfield, Mass., has published a pamphlet: *Elementary Algebra Outline based upon College Entrance Requirements and Examination Papers* (New York, Newson and Co., 1922, 48 pages).

A twenty-nine page printed report to the College Entrance Examination Board upon Elementary Algebra, Advanced Algebra, and Plane Trigonometry by the Commission on College Entrance Requirements in Mathematics, 431 West 117th Street, New York City, has recently been distributed. The Commission's Report on Plane Geometry and Solid Geometry is being printed separately. The report is elaborate and specific and should come to the attention of all interested in the ground covered in sub-freshman work in mathematics. Professor W. F. OSGOOD, of Harvard University, is chairman of the Commission.

We have already indicated (*1921*, 267) the contents of the first edition of

Suggestions for Students of Mathematics. Mathematics and Life Activities (7 pages), issued in March, 1921, by the Department of Mathematics of Brown University. An extended review appeared in *Mathematics Teacher*, volume 14, pages 349–350. A third revised and enlarged edition was published in October, 1922. In this edition a new section is devoted to a "Literature List"; that is, a list of 23 books, in English, and in the mathematical library of the University, "which may be read with profit—though not in every case with complete understanding—by those who are taking, or have credit for, freshman mathematics." A limited number of these pamphlets are available for distribution to those interested.

Two new parts of the *Encyklopädie der Mathematischen Wissenschaften* have been published April 1 and May 15, 1922, respectively. The first part, V-2-5, "Elektronentheorie der Metalle" by Rudolf Seeliger, 15+777-878 pages, completes volume V-2. The second part, III-1-9, "abgeschlossen 31 Aug. 1916", is "Polyeder und Raumeinteilungen" by Ernst Steinmetz, 1-139, special paging in italics. It is announced that III-1-8 with register for III-1 will appear later. There are now 10 complete part volumes of the *Encyklopädie*, namely: I-1, I-2, II-1-1, II-1-2, II-2, IV-1, IV-3, IV-4, V-1, and V-2.

For many years, 1900-1915, an important feature of *Bibliotheca Mathematica* was a section devoted to "Kleine Bemerkungen" on the latest edition of Cantor's *Vorlesungen über Geschichte der Mathematik*. In this way a great many corrections, and much valuable information, were brought together. In *Jahresbericht der deutschen Mathematiker-Vereinigung*, volume 31, 1922, pages *73-77*, a section of this kind has been started, and will continue so long as the publication of *Bibliotheca Mathematica* is suspended. In this section there are 11 notes, by Ferdinand Rudio, on various passages in Cantor's volume 3, pages 549-624.

ARTICLES IN CURRENT PERIODICALS.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 28, January–February, 1922: "The twenty-eighth summer meeting of the American Mathematical Society" by R. G. D. Richardson, 1–15; A letter from the President of the Society, G. A. Bliss, 16; "Report on topics in the theory of divergent series" by W. A. Hurwitz, 17–36; "Note on an irregular expansion problem" by D. Jackson, 37–41; "The Bôcher Memorial Prize" by E. B. Van Vleck, 42–44; "On Kakeya's minimum area problem" by W. B. Ford, 45–53; "Convergence of sequences of linear operations" by T. H. Hildebrandt, 53–58; Review by D. Jackson of C. de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable Réelle* (Paris, 1919), 59–61; Reviews by D. E. Smith of B. Lefebvre, *Notes d'Histoire des Mathématiques (Antiquité et Moyen Âge)* (Bruxelles, 1920), of *Bibliotheca Chemico-Mathematica* (2 volumes, London, 1921), and of W. W. Bryant, *Kepler* (London, 1920), by P. Field of P. Burgatti, *Lezioni di Meccanica Razionale* (2d edition, Bologna, 1919), and of C. Burali-Forti and I. Boggio, *Meccanica Razionale* (Turin and Genoa, 1921), by A. R. Crathorne of A. Loewy, *Mathematik des Geld- und Zahlungsverkehrs* (Leipzig and Berlin, 1920), by V. Snyder of T. Schmid, *Darstellende Geometrie*, Vol. II (Berlin and Leipzig, 1921), by H. B. Phillips of F. Reiche, *Die Quantentheorie, ihr Ursprung und ihre Entwicklung* (Berlin, 1921), and by D. N. Lehmer of L. Poletti, *Tavole di Numeri Primi entro Limite Diversi, e Tavole Affini* (Milan, 1920), 62–71; Notes, 72–82; New Publications, 83–88—March: "The October meeting of the American Mathematical Society" by R. G. D. Richardson, 89–94; "The October meeting of the San Francisco Section" by B. A. Bernstein, 94–97; "The simple group of order 2520" by G. A. Miller, 98–102; "A theorem of oscillation" by W. E. Milne, 102–104; "A two-way infinite series for Lebesgue integrals" by M. B. Porter, 105–108; "Note on Euler's

φ-function" by R. D. Carmichael, 109–110; "Extensions of Dirichlet multiplication and Dedekind inversion" by E. T. Bell, 111–122; "Charles Leonard Bouton" by W. F. Osgood, J. L. Coolidge, G. H. Chase, 123–124 [A Minute read before the Faculty of Harvard University, March 28, 1922]; Review by V. Snyder of *Felix Klein: Gesammelte mathematische Abhandlungen.* Vol. 1. *Liniengeometrie, Grundlegung der Geometrie, Zum Erlanger Programm* (Berlin, 1921), 125–129; Reviews by A. Emch of G. Scheffers, *Lehrbuch der darstellenden Geometrie* (2 volumes, Berlin, 1919, 1920), and of G. B. Halsted, *Girolamo Saccheri's Euclides Vindicatus* (Chicago, 1920), by L. W. Dowling of E. Bally, *Géométrie Synthétique des Unicursales de Troisième Classe et de Quatrième Ordre* (Paris, 1920), and by F. Cajori of A-C. Clairaut, *Éléments de Géométrie* (Paris, 1920), 130–133; Notes, 134–138; New Publications, 139–144—April–May: "The fourteenth regular meeting of the Southwestern Section" by E. B. Stouffer, 145–147; "The twenty-eighth annual meeting of the American Mathematical Society" by R. G. D. Richardson, 148–164; "The moment of inertia in the problem of n bodies" by W. D. MacMillan, 165–168; "Substitutions commutative with every substitution of an intransitive group" by G. A. Miller, 168–170; "Summable infinite determinants" by W. L. Hart, 171–178; "Algebraic guides to transcendental problems" by R. D. Carmichael, 179–210; Review by J. P. C. Southall of *Oeuvres complètes de Christiaan Huygens publiées par la Société Hollandaise des Sciences.* Tome treizième. *Dioptrique* (La Haye, 1916), 211–214; "Books on relativity" by G. D. Birkhoff, 215–221 [Review of *Das Relativitätsprinzip. Lorentz. Einstein. Minkowski* (Leipzig and Berlin, 1920), H. Weyl, *Raum. Zeit. Materie* (Berlin, 1921), A. Einstein, *Relativity. The special and the general Theory* (Translated by R. W. Lawson, New York, 1921), R. D. Carmichael, *The Theory of Relativity* (New York, 1920), A. Angerbach, *Das Relativitätsprinzip* (Leipzig and Berlin, 1920), A. N. Whitehead, *The Concept of Nature* (Cambridge, England, 1920), and L. E. J. Brouwer, *Wiskunde, Waarheid, Werkelijkheid* (Groningen, 1919)]; Reviews by E. B. Wilson of C. Schaefer, *Einführung in die theoretische Physik* (Berlin, 1921), by E. S. Allen of D. M. Y. Sommerville, *The Elements of Non-Euclidean Geometry* (Chicago and London, 1919), by E. W. Brown of *Annuaire du Bureau des Longitudes pour 1921* (Paris, 1921), by J. W. Young of L. Heffter, *Die Grundlagen der Geometrie als Unterbau für die Analytische Geometrie* (Leipzig and Berlin, 1921), by C. N. Reynolds of L. Fabre, *Les Théories d'Einstein* (Paris, 1921), 222–224; Notes, 225–227; New Publications, 228–232.

Bulletin des Sciences Mathématiques, volume 46, May, 1922: "Sur l'itération de certaines fonctions algébriques" by M. P. Fatou, 188–198; "Sur un invariant cinématique et le théorème de la composition des vitesses" by M. G. Koenigs, 198–199; "Sur un théorème de M. Fatou" by M. G. Valiron, 200–208.

Journal de Mathématiques pures et appliquées, ninth series, volume 1, fasc. 2, 1922: "Les théories einsteiniennes et les principes du calcul intégral" by A. Buhl, 95–104; "La théorie de la relativité et les faits observés" by S. Zaremba, 105–139; "Sur les équations de la gravitation d'Einstein" by E. Cartan, 141–203; "Relativité restreinte et géométrie des systèmes ondulatoires" by J. Le Roux, 205–253.

Journal of the Indian Mathematical Society, volume 14, April, 1922: "Progress report" by P. V. Seshu Aiyar, 41–42; "A statistical study of some examination marks" by P. V. Seshu Aiyar and S. R. Ranganathan, 43–54; "Determinants whose elements are Eulerian, prepared Bernoullian, and other numbers" by C. Krishnamachary and M. Bheemasena Rao, 55–62; "Mathematics in India, then and now" (conclusion) by V. Sankaran, 63–67; "Some criteria for divisibility" by P. K. Das, 68–69; "Perpetual calendar for every year of the Christian era" by C. Ganapati Aiyar, 70–71; Questions and Solutions, 72–79.

Mathesis, volume 36, June, 1922: "Bibliographie du triangle et du tétraèdre" (to be continued) by J. Neuberg, 209–211; "Sur une transformation homographique d'une équation" by F. Simonart, 212–218; "Sur deux groupes de trois coniques" by J. Degueldre, 218–219; "Sur l'ellipse et le cercle de Nagel d'un triangle" (conclusion) by J. Neuberg, 220; "Application du calcul vectoriel à la détermination du couple gyroscopique" by F. Bouny, 221–225; "Remarques sur l' 'Arithmétique' de Simon Stevin" (to be continued) by H. Bosmans, 226–231; "Sur un théorème de M. d'Ocagne et son extension aux courbes gauches" by R. Goormaghtigh, 232–233; "Un théorème de M. Appell" by Ad. M., 234–236; "Nécrologie," 236–237; "Notes mathématiques," 237–241; Questions and Solutions, 242–256.

Nature, volume 109, June 17, 1922: "More books on relativity" by E. Cunningham, 770–772 [Review of W. H. V. Reade, *A Criticism of Einstein and his Problem* (Oxford, 1922), H. Dingle, *Relativity for All* (London, 1922), C. Nordmann, *Einstein and the Universe: A Popular Exposition of the Famous Theory* (London, 1922), J. Becquerel, *Le Principe de Relativité et la Théorie de la*

Gravitation (Paris, 1922), G. Mie, *La Théorie einsteinienne de la Gravitation: Essai de vulgarisation de la théorie* (Paris, 1922), E. M. Lémeray, *L'Éther actuel et ses précurseurs (simple récit)* (Paris, 1922), C. Von Horvath, *Raum und Zeit im Lichte der speziellen Relativitätstheorie* (Berlin, 1921). Chiefly a criticism of Nordmann's book. "The book is certainly readable. The language is not only clear, but also picturesque. 'Einstein may be a treasure, but there is a fearsome troop of mathematical reptiles keeping inquisitive folk away from it. Let us drive them off with the whip of simple terminology, and approach the splendour of Einstein's theory.' This is the author's intention, like that of many others; how does he succeed? . . . Now comes the task of describing in 'simple terminology' the work of reconstruction. The reader must judge whether the following is simpler than a brief algebraic equation: 'The distance in time and the distance in space are numerically to each other as the hypotenuse and another side of a rectangular triangle are to the third side, which remains invariable. Taking this third side for base, the other two will describe above it, a triangle more or less elevated according as the velocity of the observer is more or less reduced. This fixed base is a quantity independent of the velocity. It is this which Einstein has called the "*Interval*" of events.' . . . M. Nordmann is well equipped in many ways for it. But he has fallen into the very common error of supposing that the essential truth can be given while omitting the demand for concentrated thinking on vital details. The world suffers far too much from loose thinking already. Vague generalisations, misleading analogies, superficial manifestations are made to do duty for precise statements, logical reasoning, and fundamental principles. It is not necessarily true that mathematical skill is the only way of approach to an understanding of Einstein's fundamental ideas. But it is certain that if such an understanding is to be reached it can only be by going down to a patient analysis of our own preconceived notions until we find them insufficient."]; Review of L. E. Dickson, *First Course in the Theory of Equations*, 773–774 ["Prof. Dickson's book possesses all the merits of an excellent text-book, and it is to be hoped that its circulation will be a wide one."]; "Immediate solution of dynamical problems" by G. Greenhill, 778 ["A discussion is submitted here in the manner called elementary, where the theorems of the gravitation of a sphere are proved for any portion of a spherical surface, such as a bowl, before proceeding to the result for a complete sphere."]—June 24: "A century of astronomy" by A. S. Eddington, 815–817 [From the presidential address delivered before the Royal Astronomical Society on May 30.]—Volume 110, July 1: "Advanced mathematical study and research at Cambridge" by H. S. Carslaw, 8—July 8: "Metric and British measures" by R. J. T., 29–30 ["In view of the vigorous and sustained efforts of the exponents of the metric system, and the eminent names that are to be found among them, it is perhaps not a little surprising that it makes so little progress towards general acceptance in Great Britain. . . . A fundamental distinction must, at the very outset, be drawn between the importance of stability in the units of quantity and of dimension respectively. The units of mass and capacity, speaking generally, serve simply for determining a certain quantity of goods, and the margin of tolerance is usually fairly large; . . . The unit of length is of a different character. Size, which determines the interchangeability of parts and fittings, is not capable of ready adjustment, and an error in dimensions often involves the waste of the whole article. . . . The view is quite widely held that the Imperial units are, as magnitudes, more suitable for commercial purposes than the metric. There is, therefore, at least a possibility that the solution of the metric controversy may be found in the development of a system based upon the British units, but so modified as to be capable of treatment on pure decimal lines. The Report of the Committee of the Conjoint Board suggests that the possibilities of such a solution should be explored, and one experiment in this direction has already been tried with success. The troy pound was abolished in 1878, but the troy ounce was too firmly established to be dismissed entirely. Trade in the precious metals, however, is now carried on in terms of troy ounces only, and bullion weights are made up solely in decimal multiples and sub-multiples of that unit. There has certainly been some activity in this direction in recent years, and should a really logical system upon a decimal basis be devised and secure general acceptance in the countries now using Imperial units, it may be found that these units are, after all, destined to survive."]; "Absolute measurements of sound" by A. G. Webster, 42–45; "Prof. J. C. Kapteyn, For. Mem., R. S." by A. C. D. Crommelin, 48–49.

Revue générale des Sciences, volume 33, May 30, 1922: "A propos de la conception einsteinienne de l'espace fini" by R. Paucot, 289–290—June 15: "Une famille d'astronomes: les Herschel. A propos du centenaire de la mort de William Herschel" by E. Doublet, 326–330.

Revue de Mathématiques Spéciales, volume 32, July, 1922: "Démonstration géométrique des formules de Serret-Frenet" by L. Pomey, 217–219.

SCIENCE, volume 55, June 23, 1922: "Are scientists encouraging popular ignorance?" by E. C. Bingham, 664–667 [Considers the objections to the metric system in the report of the National Industrial Conference Board.]—Volume 56, July 14: "Methods of German publishers" by R. C. Archibald, 45 ["In Germany the chief publishers of mathematical books and periodicals are Springer, Teubner, and Vereinigung wissenschaftlicher Verleger (a combination of the firms: Göschen, Guttentag, Reimer, Trübner and Veit). They have decided that for their mathematical publications of 1922 Americans shall, in general, be required to pay at least as much as $2.40 per 100 marks of the price for Germany. Of *Jahrbuch über die Fortschritte der Mathematik*, volume 45 . . . part 1 . . . and part 2 . . . are sold in Germany for 73 and 190 marks respectively. The corresponding prices for America are $4.65 and $9.00! Such extortion ought appreciably to hasten the appearance of an American abstract journal, the establishment of which has been already approved by the National Research Council. But again, *Journal für die reine und angewandte Mathematik* (Crelle), volume 151 (1920–21), is sold in Germany for 96 marks; the price to America is $6.00! . . . The above facts, obtained from the publishers themselves on May 26 and May 31, 1922, will probably suggest to mathematicians the immediate cancellation of all contemplated orders for the publications of Vereinigung wissenschaftlicher Verleger—at least."]

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

2981. Proposed by H. P. MANNING, Providence, R. I.

Find the envelope of the circle on which two diametrically opposite points divide in a given ratio the focal radii of a variable point on an ellipse or hyperbola.

2982. Proposed by H. P. MANNING, Providence, R. I.

Solve the following problem without using analytic geometry: A triangular yard has a post at each corner. Given the lengths of the sides of the yard and of the posts, find the length and the position of the foot of a ladder that will just reach the top of each post without changing this position.

2983. Proposed by R. S. UNDERWOOD, Alabama Polytechnic Institute.

Prove that the following terminating series have the sums indicated, where by 0! is meant 1:

(1) $\dfrac{1}{n!\,1!}+\dfrac{1}{(n-2)!\,3!}+\dfrac{1}{(n-4)!\,5!}+\cdots=\dfrac{2^n}{(n+1)!}$,

(2) $\dfrac{1}{(n+1)!\,0!}+\dfrac{1}{(n-1)!\,2!}+\dfrac{1}{(n-3)!\,4!}+\cdots=\dfrac{2^n}{(n+1)!}$.

(These coincide for n even but are distinct for n odd),

(3) $\dfrac{1}{n!\,1!}-\dfrac{1}{(n-2)!\,3!}+\dfrac{1}{(n-4)!\,5!}-\cdots=\dfrac{\alpha(-4)^{[n/4]}}{(n+1)!}$

where $[n/4]$ denotes the greatest integer not exceeding $n/4$, and where $\alpha = 1, 2, 2, 0$ according as n is congruent to 0, 1, 2, 3 (mod 4).

Obtain a similar form for the series

(4) $\dfrac{1}{(n+1)!\,0!}-\dfrac{1}{(n-1)!\,2!}+\dfrac{1}{(n-3)!\,4!}-\cdots$.

Obtain also the sums of the following terminating series and of the four series obtained from them by changing the signs of the alternate terms:

(5) $\frac{1}{n!\,1!} + \frac{1}{(n-4)!\,5!} + \frac{1}{(n-8)!\,9!} + \cdots,$

(6) $\frac{1}{(n+1)!\,0!} + \frac{1}{(n-3)!\,4!} + \frac{1}{(n-7)!\,8!} + \cdots,$

(7) $\frac{1}{(n-2)!\,3!} + \frac{1}{(n-6)!\,7!} + \frac{1}{(n-10)!\,11!} + \cdots,$

(8) $\frac{1}{(n-1)!\,2!} + \frac{1}{(n-5)!\,6!} + \frac{1}{(n-9)!\,10!} + \cdots.$

2984. Proposed by A. L. CANDY, University of Nebraska.

Find the number of numbers of n digits each that can be written with n consecutive digits, allowing all possible repetitions, such that the sum of the digits in each number is a multiple of n.

2985. Proposed by A. L. CANDY, University of Nebraska.

Find the number of combinations of n digits each that can be made with the first n consecutive digits, allowing repetitions, and such that the sum of the digits in each combination is a multiple of n.

SOLUTIONS.

2890 [1921, 184]. Proposed by B. F. FINKEL, Drury College.

Having given a triangle whose base is $2c$ and (a) the sum of whose other two sides is $2a$, (b) the difference of whose other two sides is $2a$, determine the envelope of the perpendicular bisectors of the variable sides.

I. Solution by the Proposer.

(a) Letting the vertices of the triangle be $(-c, 0)$, $(c, 0)$ and (α, β), we have

$$\alpha^2/a^2 + \beta^2/b^2 = 1, \tag{1}$$

where $b^2 = a^2 - c^2$.

The equation of the perpendicular bisector of the line connecting $(c, 0)$ and (α, β) may be written[1]

$$(\alpha - x)^2 + (\beta - y)^2 = (x - c)^2 + y^2. \tag{2}$$

Taking α as a variable parameter and differentiating with respect to α, we have $d\beta/d\alpha = -b^2\alpha/a^2\beta = -(\alpha - x)/(\beta - y)$, and our problem is to eliminate α and β.

The elimination may be carried out as follows:

$$\beta = b^2\alpha y/[b^2\alpha - a^2(\alpha - x)],$$

and if we write e for c/a, we have

$$\beta = \frac{(1 - e^2)\alpha y}{x - e^2\alpha} \qquad \text{and} \qquad \beta - y = \frac{(\alpha - x)y}{x - e^2\alpha},$$

and equations (1) and (2) become

$$a^2 - \alpha^2 = \frac{(1 - e^2)\alpha^2 y^2}{(x - e^2\alpha)^2}, \tag{3}$$

and

$$(x - ae)^2 + y^2 - (\alpha - x)^2 = \frac{(\alpha - x)^2 y^2}{(x - e^2\alpha)^2}; \tag{4}$$

whence

$$\frac{(x - ae)^2 + y^2 - (\alpha - x)^2}{a^2 - \alpha^2} = \frac{(\alpha - x)^2}{(1 - e^2)\alpha^2},$$

and therefore

$$(a^2 - e^2\alpha^2)(\alpha - x)^2 = (1 - e^2)[(x - ae)^2 + y^2]\alpha^2. \tag{5}$$

Now equation (3) is the same as

$$(a^2 - \alpha^2)(e^2\alpha - x)^2 = (1 - e^2)\alpha^2 y^2, \tag{6}$$

[1] It should be noticed that this solution, as well as the next, gives only one part of the locus, the two parts being symmetrical with respect to the y-axis—Editors.

and if we subtract and divide by $1 - e^2$ we get

$$(1 + e^2)a^2\alpha^2 - 2a^2x\alpha - e^2\alpha^4 + x^2\alpha^2 = (x - ae)^2\alpha^2,$$

or simplifying further and dividing also by $e\alpha - a$,

$$e\alpha^2 + a\alpha - 2ax = 0;$$

whence

$$x = \frac{(a + e\alpha)\alpha}{2a}. \tag{7}$$

Equation (6) then gives

$$y = \pm \frac{e\alpha + a - 2ae^2}{2a} \sqrt{\frac{a^2 - \alpha^2}{1 - e^2}}. \tag{8}$$

These are the parametric equations of the locus.

(*b*) The parametric equations for this part of the problem may be obtained in the same way.

II. Solution by J. K. Whittemore, Yale University.

The fixed vertices of the triangle are the foci of (*a*) an ellipse or (*b*) a hyperbola described by the variable vertex; the variable sides are the focal radii of the ellipse or hyperbola; the middle point of each focal radius describes a similar conic, the center of similitude being the corresponding focus. The problem stated is a special case of the following:

Given any curve whose polar equation is $r = f(\varphi)$ at each point P of the curve a line PQ is constructed perpendicular to OP, the radius vector of P. The envelope of PQ is required.

If the length PQ is denoted by p the rectangular coördinates of Q are

$$x = r\cos\varphi - p\sin\varphi, \qquad y = r\sin\varphi + p\cos\varphi.$$

In order that the locus of Q be the envelope of PQ it is necessary and sufficient that

$$\frac{dy}{dx} = -\cot\varphi$$

or

$$\frac{(r' - p)\sin\varphi + (r + p')\cos\varphi}{(r' - p)\cos\varphi - (r + p')\sin\varphi} = -\frac{\cos\varphi}{\sin\varphi},$$

where accents denote differentiation with respect to φ. From the last equation $p = r'$, so that the equations of the envelope in terms of the parameter φ are

$$x = r\cos\varphi - r'\sin\varphi, \qquad y = r\sin\varphi + r'\cos\varphi.$$

In the problem stated the envelope is given by substituting

$$r = f(\varphi) = \frac{1}{2}\frac{me}{1 + e\cos\varphi}, \qquad e = \frac{c}{a}, \qquad m = \pm\frac{a^2 - c^2}{c},$$

where the upper and lower signs in m correspond to the ellipse and hyperbola respectively.[1]

We may also give the intrinsic equation of the envelope in the general case. From its equations

$$dx = -(r'' + r)\sin\varphi d\varphi, \qquad dy = (r'' + r)\cos\varphi d\varphi,$$

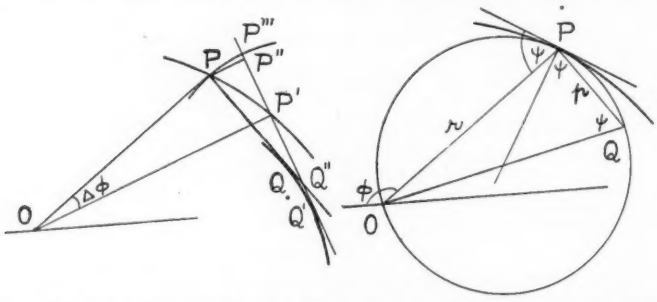
$$\rho = \frac{ds}{d\varphi} = r'' + r.$$

If the envelope is given in intrinsic form, $\rho = F(\varphi)$, the original or pedal curve is given by solving the differential equation, $r'' + r = F(\varphi)$. For example, (1) if the envelope is a point $F(\varphi) = 0$, $r = A\cos(\varphi - \alpha)$, and the pedal curve is a circle passing through the origin; (2) if the envelope is a circle $F(\varphi) = A$, $r = A + B\cos(\varphi - \alpha)$, and the pedal curve is a limaçon. Both of these results are easily verified.

Note by Otto Dunkel, Washington University—The results obtained by Professor

[1] The equations obtained in this way reduce to those given under I if we express $\cos\varphi$ and $\sin\varphi$ in terms of α and transfer the origin to the center—Editors.

Whittemore may be obtained geometrically as follows: Consider two neighboring points, P and P', on the given curve and let O be the pole, Q and Q' the points of tangency of the perpendiculars with their envelope. Then the intersection Q'' of PQ and $P'Q'$ lies on the circle through



O, P and P'. When P' approaches P as a limit the circle has for its limit a circle tangent to the given curve at P and passing through O. This gives a geometrical construction for Q. If ψ is the angle between OP and the tangent to the given curve then $\angle OQP = \psi$ and $\tan\psi = r/p$. But the well-known formula of polar coördnaites gives $\tan\psi = r/(dr/d\varphi)$ and hence $p = dr/d\varphi$. Consider that involute of the envelope (Q) which passes through P and let it cut $Q'P'$ in P'''; let P'' be the foot of the perpendicular from P upon $Q'P'$. Then $P'''Q' - PQ = \Delta s$ and $P'''Q' = P'''P'' + r\sin\Delta\varphi + p + \Delta p$ and hence

$$\frac{P'''P''}{\Delta\varphi} + \frac{r\sin\Delta\varphi}{\Delta\varphi} + \frac{\Delta p}{\Delta\varphi} = \frac{\Delta s}{\Delta\varphi}.$$

On taking the limit we have

$$r + \frac{dp}{d\varphi} = \rho,$$

since $P'''P''$ is equal to the length of the perpendicular from P to the tangent to the involute at P''', and hence it is an infinitesimal of the second order.

2892 [1921, 184]. Proposed by R. T. McGREGOR, Bangor, Calif.

Two parabolas have parallel axes. Prove that their common chord bisects their common tangent.

I. Solution by Marcia L. Latham, Hunter College.

Let the axes be rectangular, with the x-axis parallel to the axes of the parabolas; let P_1 and P_2, respectively, be the points of contact of the common tangent with the two parabolas, and P_3, the midpoint of P_1P_2.

Then the equations of the parabolas will be

$$y^2 + 2B_1x + 2C_1y + D_1 = 0, \tag{1}$$
$$y^2 + 2B_2x + 2C_2y + D_2 = 0. \tag{2}$$

The tangent to (1) at P_1 is

$$yy_1 + B_1(x + x_1) + C_1(y + y_1) + D_1 = 0.$$

But this passes through P_2; therefore,

$$y_1y_2 + B_1(x_1 + x_2) + C_1(y_1 + y_2) + D_1 = 0. \tag{3}$$

Again the tangent to (2) at P_2 passes through P_1; therefore,

$$y_1y_2 + B_2(x_1 + x_2) + C_2(y_1 + y_2) + D_2 = 0. \tag{4}$$

Subtracting (4) from (3), we can write

$$2(B_1 - B_2)\frac{x_1 + x_2}{2} + 2(C_1 - C_2)\frac{y_1 + y_2}{2} + (D_1 - D_2) = 0. \tag{5}$$

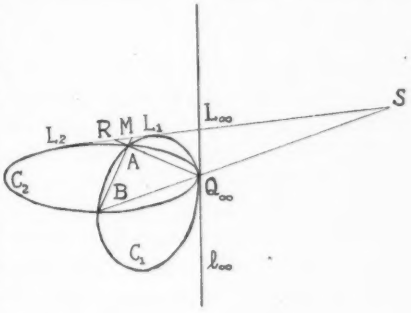
But the equation of the common chord is

$$2(B_1 - B_2)x + 2(C_1 - C_2)y + (D_1 - D_2) = 0, \tag{6}$$

and thus (5) states that the coördinates of P_3 satisfy (6) and the common chord bisects the common tangent.[1]

II. Solution by W. E. Cleland, Princeton University.

Let the two conics C_1 and C_2 be tangent to the line at infinity of a Euclidean plane at Q_∞. Then C_1 and C_2 are parabolas whose axes are parallel. Let a common tangent be L_1L_2, tangent to C_1 at L_1 and to C_2 at L_2, and intersecting the line at infinity in L_∞. The common chord is AB, intersecting L_1L_2 in M. Let the intersections of AQ_∞ and BQ_∞ with L_1L_2 be R and S, respectively.



Since the line L_1L_2 is tangent to C_1 at L_1, L_1 is a double point in the involution determined on L_1L_2 by the pairs of points $R\,S$, and $M\,L_\infty$. Similarly, L_2 is the other double point of the same involution. Hence M is the harmonic conjugate of L_∞ with respect to L_1 and L_2. M is therefore the midpoint of L_1L_2. (For proofs of these statements see Veblen and Young, *Projective Geometry.*)

Also solved by William Hoover, L. C. Mathewson, J. B. Reynolds, J. K. Whittemore, and F. L. Wilmer.

NOTES AND NEWS.

It is hoped that readers of the Monthly will coöperate in contributing to the general interest of this department by sending items to H. P. Manning, 69 Weymouth St., Providence, R. I.

Charles Albert Fischer, Seabury professor of mathematics at Trinity College, Hartford, Conn., died December 7,[2] 1922. He was born at Wheaton, Ill., October 3, 1884, and received the degree of A.B. from Wheaton (Ill.) College in 1905. He spent 1909–1912 in graduate study in mathematics and astronomy at the University of Illinois (A.M., 1910) and University of Chicago (Ph.D., 1912). His thesis, published in the *American Journal of Mathematics*, 1913, was entitled: "A generalization of Volterra's derivative of a function of a curve." At the meeting of the American Mathematical Society at the University of Rochester last September he gave, by invitation, an hour lecture on "Functions of lines." He was a charter member of the Association.

Arnold Sommerfeld, professor of mathematical physics at the University of Munich, will be in residence at the University of Wisconsin for the first semester of the academic year 1922–1923, holding the Karl Schurz Memorial Professorship in the University for that period. Professor Sommerfeld is expected to give a

[1] The solution will have a particularly simple form if we take the common tangent and the line through its mid-point parallel to the axes of the parabolas for coördinate axes, giving us a system of oblique coördinates. The common chord will then appear as a line through the origin —Editors.

[2] The date is given incorrectly as December 9 in *Science*, December 15.

three-hour course on atomic structure and a second three-hour course either on the analysis of wave propagation, or in general theory of relativity. Professor Sommerfeld is one of the outstanding scholars in the field of mathematical physics in Europe. His contributions to this field are well known to American scholars. The Karl Schurz Memorial Professorship in the University of Wisconsin was founded in 1910 as an exchange professorship with the German universities. The appointment of Professor Sommerfeld marks its resumption after the interruption caused by the war.

At an informal dinner of the Association of Teachers of Mathematics in New England, in Riverbank Court, Cambridge, on October 21, 1922, W. L. VOXBURGH, of the Boston Normal School, spoke on "Mathematics in California" and G. W. EVANS, of the Charlestown High School, on "Mathematics in China."

At the meetings of the American Mathematical Society at the University of Rochester, September 7–8, 29 papers by the following mathematicians were presented: C. C. CAMP, ELIZABETH CARLSON, E. H. CLARKE, F. N. COLE, A. H. COWLING, E. L. DODD, L. P. EISENHART, H. J. ETTLINGER, G. C. EVANS (2), C. A. FISCHER, P. FRANKLIN, W. C. GRAUSTEIN, O. D. KELLOGG, S. LEFSCHETZ, N. J. LENNES, G. A. MILLER, C. N. MOORE, R. L. MOORE and J. R. KLINE, F. H. MURRAY (2), A. MYLLER, K. SCHMIDT, I. J. SCHWATT (3), F. R. SHARPE and V. SNYDER, W. L. G. WILLIAMS, S. D. ZELDIN.

At the celebration of the seven hundredth anniversary of the founding of the University of Padua, May 14–17, 1922, twenty-three delegates from the United States were present. These included the following professors of astronomy and mathematics: R. G. AITKEN, of the Lick Observatory and the University of California, R. C. ARCHIBALD, of Brown University, JOSEPH LIPKA, of Massachusetts Institute of Technology, and VIRGIL SNYDER, of Cornell University. On each of these delegates the University of Padua conferred the honorary degree of doctor. The two delegates from the Dominion of Canada were Professor ARCHIBALD, representing Mount Allison University, and the late Dr. GEORGE PARKIN, the organizer of the Rhodes Scholarships, representing the University of New Brunswick.

The MONTHLY would be remiss in its duty if it did not from time to time call attention to special opportunities in the mathematical field. The United States Civil Service Commission states that there is urgent need for eligibles to fill positions of computer in the U. S. Coast and Geodetic Survey. The basic salaries are $1,400 a year for positions in Washington, D. C., and $2,000 a year for positions in Manila, P. I. In addition to the basic salaries the increase of $20 a month granted by Congress will be allowed to those whose services are found satisfactory. Full information and application blanks may be secured from the United States Civil Service Commission, Washington.[1]

[1] A circular dated November 29, 1922, announces an open competitive examination for computer March 7 and 8, 1923.

Published February 13, 1923.

CONTRADICTIONS IN THE LITERATURE OF GROUP THEORY.

By G. A. MILLER, University of Illinois.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE MATHEMATICAL ASSOCIATION OF AMERICA, SEPTEMBER 6, 1922.

The ancient Greeks, even before the time of Euclid, assumed that if two hypotheses lead to contradictory results they cannot both be true, and in Euclid's *Elements* frequent use is made of a method of proof known as *reductio ad absurdum*, called by the Greeks "reduction to the impossible." The "impossible" here often meant a result which contradicted a result which was assumed to be well established. While contradictions are so distasteful to the normal mind they have not been completely avoided in the mathematical literature. For instance, even Euclid used the term circle both for a curve and also for the area inclosed by this curve, and the double meaning of this particular term has persisted to our day. The seriousness of contradictions in the literature of our subject is illustrated by following well-known dictum due to Poincaré: "Mathematics is the art of giving the same name to different things." [1]

Many contradictions have been so mellowed by age that they are scarcely noticed by the teacher while the student is greatly perplexed by them. As teachers we cannot be too careful to point out emphatically where such contradictions appear. While this might seem to be a sufficient reason for the choice of the present subject for this extraordinary occasion, it may be admitted that the real reason was much more personal. A referee of a paper which I offered for publication did not agree with me in view of the fact that we used different definitions of the same term. Both of these definitions were sanctioned by long usage but it was only after considerable correspondence that the real source of the difficulty became clear. Hence I address you on this subject in somewhat the same state of mind as that exhibited by the rich man in the Bible who wanted Abraham to send Lazarus to testify to his five brethren "lest they also come into this place of torment."

The source of the particular difficulty to which I have just referred proved to be the fact that both in abstract group theory and in substitution group theory we commonly say that a given finite group is transformed into itself only by the operators or the substitutions of its holomorph. Hence some operators or substitutions transform such a group into itself while other operators or substitutions do not have this property. On the other hand, every possible operator or substitution transforms a given group into a simply isomorphic group, and all the simply isomorphic groups are usually regarded as the same group when abstract groups or regular substitution groups are listed. Hence we also say, at least implicitly, that a given group is transformed into itself by every possible group operator or by every possible substitution.

[1] *Bulletin des Sciences Mathématiques*, vol. 32, 1908, p. 174.